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Cotangent bundle quantization: entangling of metric and magnetic field

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Abstract
For manifolds $\mathcal{M}$ of noncompact type endowed with an affine connection (for example, the Levi–Civita connection) and a closed 2-form (magnetic field), we define a Hilbert algebra structure in the space $L^2(T^*\mathcal{M})$ and construct an irreducible representation of this algebra in $L^2(\mathcal{M})$. This algebra is automatically extended to polynomial in momenta functions and distributions. Under some natural conditions, this algebra is unique. The non-commutative product over $T^*\mathcal{M}$ is given by an explicit integral formula. This product is exact (not formal) and is expressed in invariant geometrical terms. Our analysis reveals that this product has a front, which is described in terms of geodesic triangles in $\mathcal{M}$. The quantization of δ-functions induces a family of symplectic reflections in $T^*\mathcal{M}$ and generates a magneto-geodesic connection Γ on $T^*\mathcal{M}$. This symplectic connection entangles, on the phase space level, the original affine structure on $\mathcal{M}$ and the magnetic field. In the classical approximation, the $\hbar^2$-part of the quantum product contains the Ricci curvature of Γ and a magneto-geodesic coupling tensor.

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1. Introduction

There is a well-known quantum product defined by Groenewold [1], Moyal [2] and Berezin [3] over the phase space $\mathbb{R}^{2n}$. This non-commutative associative product of functions corresponds to the Weyl symmetrization rule for ordering the quantum coordinates $\hat{q}^j$ and momenta $\hat{p}_k$ which obey the canonical commutation relations

$$[\hat{q}^j, \hat{q}^k] = 0, \qquad [\hat{q}^j, \hat{p}_k] = \mathrm{i}\hbar\delta^j_k, \qquad [\hat{p}_j, \hat{p}_k] = 0. \tag{1.1}$$

In the classical limit $\hbar \to 0$, this quantum product reduces to the usual product of functions over $\mathbb{R}^{2n}$ and yields the standard symplectic structure $\mathrm{d}p \wedge \mathrm{d}q$ on $\mathbb{R}^{2n}$.

There is also a magnetic analogue [4, 5] of this quantum product, where $\hat{p}$ plays the role of the kinetic momenta and satisfies the commutation relations

$$[\hat{q}^j, \hat{q}^k] = 0, \qquad [\hat{q}^j, \hat{p}_k] = \mathrm{i}\hbar\delta^j_k, \qquad [\hat{p}_j, \hat{p}_k] = \mathrm{i}\hbar F_{kj}(\hat{q}). \tag{1.2}$$

Here, F is the Faraday tensor (the strength of the magnetic field for $n = 3$). The small $\hbar$ asymptotics of this product generates the 'magnetic' symplectic structure on $\mathbb{R}^{2n}$

$$\omega = \mathrm{d}p \wedge \mathrm{d}q + \tfrac{1}{2}F(q)\,\mathrm{d}q \wedge \mathrm{d}q. \tag{1.3}$$

The magnetic algebra generated by relations (1.2) is an interesting and useful object for physical and mathematical applications [6–15]. In particular, the case of quadratic magnetic field F represents an example of quadratic quantum algebra (1.2) which corresponds to the symplectic space $\mathbb{R}^{2n}$ of constant non-zero curvature [5]. Also, it is useful to recall that the form of relations (1.2) is gauge invariant, i.e. it does not depend on the choice of magnetic potential, and so all the calculations in this algebra are gauge independent *a priori*.

One would like to examine what happens in this framework if the flat q-space $\mathbb{R}^n$ is replaced by a curved manifold $\mathcal{M}$. This means that the Euclidean metric on $\mathbb{R}^n$ is replaced by a Riemannian metric, or more generally, by an affine connection $\underline{\Gamma}$ on $\mathcal{M}$. Accordingly, the phase space $\mathbb{R}^{2n}$ is replaced by the cotangent bundle $T^*\mathcal{M}$.

The fundamental question arises: how to define a quantum product over $T^*\mathcal{M}$ which would naturally generalize the products appearing in the Euclidean cases (1.1) and (1.2) and incorporate the connection $\underline{\Gamma}$ on $\mathcal{M}$? This is an old quantization problem, which was posed by Dirac [16] and Mackey [17] and initially studied in [18–24] and other works.

The recent mathematical investigation of this problem has been carried out in [25–33] including the case where the metric and the magnetic field are both present on $\mathcal{M}$ [25, 26, 33]. There is also a large literature, beginning with the paper by Widom [34], where this problem was studied from the perspective of pseudodifferential and Fourier integral operators.

In spite of certain essential progress, there remain many significant open questions in this problem area.

First, we note that the papers cited above do not address the following questions.

How does the connection $\underline{\Gamma}$ *entangle with the magnetic field* F on $\mathcal{M}$ via the quantization process? Are $\underline{\Gamma}$ and F combined in a natural geometrical way?

This family of questions closely parallels the issues raised in Weyl's discovery of the gauge principle. In [35], Weyl constructed a connection on the configuration space which combined the Levi–Civita metric connection with the magnetic potential and was 'gauge' invariant. Subjected to Einstein's criticism [36] that the construction was incompatible with physical reality, Weyl revised the direction of his programme (and soon invented the beginnings of modern gauge theory). Nevertheless, we now think that Weyl's original intention finds very strong support in the quantization theory where one has a natural opportunity to extend configuration space $\mathcal{M}$ to the phase space $T^*\mathcal{M}$ and re-examine the 'connection problem' therein.

Secondly, although all the works dealing with the quantization problem over $T^*\mathcal{M}$ use more or less the same core idea for generalizing the Groenewold–Moyal product (just replace in the phase functions all the straight chords by geodesics), there is a wide variation in the definition of the amplitude functions. This variety of amplitudes illustrates the known phenomena of the non-uniqueness of quantization.

Even in the Euclidean example, an aspect of this non-uniqueness is present and correlates, for instance, with the ordering problem, cf [4, 37, 38]. In the Euclidean case, conditions which uniquely identify the Groenewold–Moyal product and Weyl ordering are known [39–41]; in the context of formal deformation theory, see the discussion in [42–44]. The main idea in all these approaches is to exploit certain symmetry group actions. For inhomogeneous manifolds $\mathcal{M}$, this is not possible. So, the question remains open: how to select a *unique quantization* on $\mathcal{M}$?

Thirdly, one may claim that in the literature there is still no explicit formula for the quantum product over $T^*\mathcal{M}$, even for the simplest examples of curved manifolds $\mathcal{M}$, even with no magnetic field.

We mean here an *exact formula*, not a formal deformation one. Such an exact formula could be applied, for instance, to highly oscillating or singularly concentrated (as $\hbar \to 0$) functions on $T^*\mathcal{M}$ which are required to describe Schrödinger quantum dynamics or eigenfunction problems on $\mathcal{M}$. On this topic, we recall the asymptotic quantization theory [45] which allows this type of 'semiclassical' $\hbar$-dependence in its symbols and deals with symplectic manifolds of general type without having a global polarization. However, for symplectic manifolds, $\prec T^*\mathcal{M}, \omega\succ$, it is natural to ask more: namely to obtain an exact, not semiclassical, quantization formula which is globally and geometrically stated on $T^*\mathcal{M}$.

In this paper, we present solutions to these questions in the case where the configuration manifold $\mathcal{M}$ is geodesically simply connected. As an example, one can take $\mathcal{M}$ to be a symmetric Riemannian manifold of noncompact type, say, the hyperboloid in the Minkowski space and, in particular, the Lobachevski plane. Another class of examples is given by manifolds $\mathcal{M} \approx \mathbb{R}^n$ whose metric is a deformation of the Euclidean one.

Part of the results described below can also be applied to generic curved manifolds $\mathcal{M}$, for instance to compact manifolds.

The magnetic field F can be an arbitrary closed 2-form on $\mathcal{M}$.

By using the averaging of F along geodesics, we define symplectic transformations of the phase space $\prec T^*\mathcal{M}, \omega\succ$ which correspond to autoparallel vector fields on $\mathcal{M}$. This is a *magneto-geodesic analogue of the Galilei translations in* $\mathbb{R}^n$. We introduce unitary operators in $L^2(\mathcal{M})$ corresponding to these symplectic transformations and exploit them to select in a unique way the quantization operation

$$f \to \hat{f} \tag{1.4}$$

on a function space over $T^*\mathcal{M}$ (section 3).

The mapping (1.4) determines an exact irreducible representation of the Hilbert algebra $L^2(T^*\mathcal{M})$ in the Hilbert space $L^2(\mathcal{M})$. We also extend this mapping to a wider algebra which includes, in particular, functions on $T^*\mathcal{M}$ polynomial in momenta, some exponential highly oscillating functions as $\hbar \to 0$, delta functions, etc. Note that we are employing here the Hilbert algebra approach to quantization theory. If one examines the C^* algebra corresponding to our quantum Hilbert algebra over $T^*\mathcal{M}$, then it is of the strict quantization type [46].

At the next stage in section 4, we analyse which symplectic transformations σ_x of the phase space $\prec T^*\mathcal{M}, \omega\succ$ correspond to the quantum δ-functions, $\widehat{\delta_x}$. In this way, a family of *magneto-geodesic reflections* on the space $T^*\mathcal{M}$ is obtained. They are the phase space analogues of the geodesic reflections in $\mathcal{M}$ interacting with the magnetic field.

In section 5, we use the related σ-reflective curves to represent the quantum product $\star$ over $T^*\mathcal{M}$ which corresponds to the quantization operation (1.4) in the usual way

$$\hat{f}\hat{g} = \widehat{f \star g}. \tag{1.5}$$

The product $\star$ is given by an exact, explicit and geometrically invariant integral formula (5.16).

This formula can be used for different subalgebras of functions over $T^*\mathcal{M}$. Being restricted to the subalgebra of polynomial in momenta functions, this formula works for the case of a generic affine manifold $\mathcal{M}$ (possibly not geodesically simply connected).

In section 6, we prove that the asymptotic expansion of the exact quantum product as $\hbar \to 0$ has the following form:

$$f \star g = fg - \frac{i\hbar}{2} f \langle \overleftarrow{\nabla} \Psi \overrightarrow{\nabla} \rangle g - \frac{\hbar^2}{8} f [\langle \overleftarrow{\nabla} \Psi \overrightarrow{\nabla} \rangle^2 + 3 \langle \overleftarrow{\nabla} \Psi \mathcal{R} \Psi \overrightarrow{\nabla} \rangle] g + O(\hbar^3). \tag{1.6}$$

Here,

$$\Psi = \begin{bmatrix} 0 & -I \\ I & F \end{bmatrix}$$

is the Poisson tensor on $T^*\mathcal{M}$ associated with the symplectic structure (1.3), ∇ denotes the covariant derivative corresponding to a symplectic connection Γ on $\prec T^*\mathcal{M}, \omega \succ$ defined by magneto-geodesic reflections (cf (6.5)) and $\mathcal{R}$ is the Ricci tensor of this connection. The phase space covariant derivative ∇ appearing in the asymptotic expansion (1.6) matches our quantization formulae with the deformation quantization [18, 22, 28, 29, 42, 47–50]. In the deformation quantization framework, the symplectic connection corresponding to a given star product is determined by the $\hbar^2$-term via a formula like (1.6).

In our approach, the connection Γ is derived in a different way, via σ-reflections. In section 6, the explicit formulae are obtained for the connection Γ and for its curvature in terms of $\underline{\Gamma}$ and F. These formulae entangle the configuration space data $\underline{\Gamma}$ and F on the phase space level. We call Γ a *magneto-geodesic connection*.

The part of Γ which depends on the magnetic field F we call a magneto-geodesic coupling. This is a 3-tensor on $\mathcal{M}$. In the case of a Riemannian manifold $\mathcal{M}$ with a Levi–Civita connection $\underline{\nabla}$, this tensor is equivalent to the one which arises in the inhomogeneous Maxwell equation (with current and charge).

The preprint version of this paper is found in quant-ph/0505144.

2. Preliminary definitions and notations

After von Neumann [51], Wigner [52], Groenewold [1] and Stratonovich [53], it was understood that the basic object of the quantization theory is a Hilbert algebra together with its exact irreducible representation in a Hilbert space.

By definition (see, for instance, in [54]), the Hilbert algebra $\mathcal{L}$ is a complete linear space with three structures: an associative product $\star$, a scalar product $(\cdot, \cdot)$ and an involution *, which are mutually consistent.

Let $\mathcal{H}$ be a Hilbert space. Then, the minimal Hilbert algebra which has an irreducible representation in $\mathcal{H}$ is the algebra of all Hilbert–Schmidt operators on $\mathcal{H}$.

The basic idea of quantization theory is to replace the operator algebra by a function algebra over an appropriate phase space. Following the correspondence principle, one is taking $\mathcal{H}$ to be

$$\mathcal{H} = L^2(\mathcal{M}, \mathrm{d}m),$$

where $\mathcal{M}$ is a configuration space, i.e. a smooth manifold with a smooth positive measure $\mathrm{d}m$. The phase space is then defined as $T^*\mathcal{M}$, i.e. the cotangent bundle over $\mathcal{M}$, and the Hilbert algebra is assumed to be

$$\mathcal{L} = L^2(T^*\mathcal{M}, \mathrm{d}l).$$

Here, $\mathrm{d}l$ is the normalized Liouville measure

$$\mathrm{d}l(x) = \frac{\mathrm{d}x}{(2\pi\hbar)^n}, \qquad \mathrm{d}x = \mathrm{d}q^1 \cdots \mathrm{d}q^n \, \mathrm{d}p_1 \cdots \mathrm{d}p_n,$$

where $x = (q, p)$, $q \in \mathcal{M}$, $p \in T_q^\star\mathcal{M}$, $n = \dim \mathcal{M}$. So, the scalar product in the algebra $\mathcal{L}$ is given by

$$(f, g) = \frac{1}{(2\pi\hbar)^n} \int_{T^*\mathcal{M}} f(x)\overline{g(x)}\, \mathrm{d}x$$

and the involution is given by the complex conjugation

$$f^* = \overline{f}.$$

In addition to the scalar product, there is the trace functional

$$\mathrm{tr}(f) \equiv \frac{1}{(2\pi\hbar)^n} \int_{T^*\mathcal{M}} f(x)\, \mathrm{d}x,$$

where $f \in L^1(T^*\mathcal{M}, \mathrm{d}l)$. We ask that the product in the algebra $\mathcal{L}$ obey the following property: the ideal $\mathcal{L}^1 \equiv \mathcal{L} \star \mathcal{L}$ is a subset of $L^1(T^*\mathcal{M}, \mathrm{d}l)$ and

$$\mathrm{tr}(f \star \overline{g}) = (f, g) \tag{2.1}$$

for any $f, g \in \mathcal{L}$.

The representation of the algebra $\mathcal{L}$ in the Hilbert space $\mathcal{H}$ is denoted by

$$f \to \hat{f} \tag{2.2}$$

and is assumed to satisfy the usual axioms

$$\hat{f}^\dagger = \widehat{\overline{f}}, \qquad \hat{f}\hat{g} = \widehat{f \star g}, \qquad \mathrm{Tr}(\hat{f}) = \mathrm{tr}(f). \tag{2.3}$$

Here, Tr indicates the operator trace and $\dagger$ the adjoint. The last axiom is restricted to the subspace $\mathcal{L}^1$.

The inverse to the mapping (2.2) is called dequantization or symbol mapping

$$\hat{f} \to f = \mathrm{Smb}(\hat{f}).$$

It is convenient to write the quantization mapping (2.2) in the integral form

$$\hat{f} = \int_{T^*\mathcal{M}} f(x)\Delta_x\, \mathrm{d}x, \tag{2.4}$$

where $\{\Delta_x\}$ is a family of operators in $\mathcal{H}$, parameterized by points $x \in T^*\mathcal{M}$. Then, the symbol mapping is given by

$$f(x) = (2\pi\hbar)^n\, \mathrm{Tr}(\hat{f}\Delta_x). \tag{2.5}$$

The first and third axioms in (2.3), together with (2.1), as well as definition (2.4), together with (2.5), are reformulated in terms of the operator family Δ as follows:

$$\Delta_x^\dagger = \Delta_x, \qquad (2\pi\hbar)^n\, \mathrm{Tr}(\Delta_x\Delta_y) = \delta_x(y), \qquad (2\pi\hbar)^n \int_{T^*\mathcal{M}} \Delta_x \otimes \Delta_x\, \mathrm{d}x = \mathcal{I}. \tag{2.6}$$

Here, δ is the Dirac delta function with respect to the canonical measure on $T^*\mathcal{M}$ and $\mathcal{I}$ is the antipodal operator $\psi \otimes \chi \to \chi \otimes \psi$ on Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

The second axiom in (2.3) reads

$$(f \star g)(x) = \int_{T^*\mathcal{M}} \int_{T^*\mathcal{M}} K_\star(x, y, z) f(y) g(z)\, \mathrm{d}y\, \mathrm{d}z, \tag{2.7}$$

where the distribution $K_\star$ is defined as

$$K_\star(x, y, z) = (2\pi\hbar)^n\, \mathrm{Tr}(\Delta_x\Delta_y\Delta_z). \tag{2.8}$$

Of course, in formulae (2.4)–(2.8), appropriate care must be taken with respect to the convergence of integrals and traces (using the weak topology and a suitable distribution extension of functions). For instance, the distribution character of $\mathrm{Tr}(\Delta_x\Delta_y)$ and $\mathrm{Tr}(\Delta_x\Delta_y\Delta_z)$ is defined by first integrating the operator-valued functions with C_0^∞ test functions and after that computing the trace.

Various symmetry properties follow directly from the definition of $K_\star$ in (2.8): it is invariant under any cyclic permutation of its arguments, and it obeys $K_\star \to \overline{K_\star}$ under the permutation of any pair of its arguments.

The family Δ was introduced by Stratonovich [53] for the case $\mathcal{M} = \mathbb{R}^n$. In [55], such a family was called a *quantizer*. See also details and examples in [56–60].

One can call the last two quantizer properties in (2.6) *orthonormality* and *operator completeness*, respectively. The identities in (2.6) imply that quantization (2.4) and dequantization (2.5) are mutually consistent.

In the next section, we construct the quantizer using the affine connection and the magnetic field on $\mathcal{M}$, and then apply formulae (2.7) and (2.8) to calculate the quantum product over $T^*\mathcal{M}$. But before that, we need to demonstrate how the $\star$-product can be extended to other classes of symbols beyond the algebra $\mathcal{L}$.

Let $\mathcal{F}$ be any subalgebra in $\mathcal{L}$. Denote by $\mathcal{F}'$ the space of linear functionals on $\mathcal{F}$. Employing the canonical measure, we identify functionals with distributions on $T^*\mathcal{M}$ via

$$\langle f, h\rangle = \int_{T^*\mathcal{M}} f(x)h(x)\,\mathrm{d}x, \qquad f \in \mathcal{F}', \quad h \in \mathcal{F}.$$

Obviously, $\mathcal{F} \subset \mathcal{F}'$. Further note that $\mathcal{F}'$ is an $\mathcal{F}$-module, i.e. $f \in \mathcal{F}', k \in \mathcal{F} \Rightarrow f \star k \in \mathcal{F}'$, $k \star f \in \mathcal{F}'$, where by definition

$$\langle f \star k, h\rangle \stackrel{\text{def}}{=} \langle f, k \star h\rangle, \qquad \langle k \star f, h\rangle \stackrel{\text{def}}{=} \langle f, h \star k\rangle, \qquad \forall h \in \mathcal{F}. \tag{2.9}$$

Denote by $\mathcal{F}_\star$ the following subset:

$$\mathcal{F}_\star = \{f \in \mathcal{F}' | f \star h \in \mathcal{F} \text{ and } h \star f \in \mathcal{F}, \forall h \in \mathcal{F}\}.$$

In particular, $\mathcal{F} \subset \mathcal{F}_\star$.

We call $\mathcal{F}$ a *normal subalgebra* if the set $\mathcal{F}_\star$ obeys the property

$$\langle f, h \star g\rangle = \langle g, f \star h\rangle, \qquad \forall f, g \in \mathcal{F}_\star, \quad \forall h \in \mathcal{F}. \tag{2.10}$$

If $\mathcal{F}$ is a normal subalgebra in $\mathcal{L} = L^2(T^*\mathcal{M})$, then the set $\mathcal{F}_\star$ is endowed with the algebra structure

$$\langle f \star g, h\rangle \stackrel{\text{def}}{=} \langle f, g \star h\rangle, \qquad \forall f, g \in \mathcal{F}_\star, \quad \forall h \in \mathcal{F}, \tag{2.11}$$

which is consistent with the involution $\overline{f \star g} = \overline{g} \star \overline{f}$.

Verification of the embedding $f \star g \in \mathcal{F}_\star$ and the $\star$-associativity is achieved by repeated applications of (2.9)–(2.11) in combination with the associativity of $\mathcal{F}$. The algebra $\mathcal{F}_\star$ is a natural extension of the subalgebra $\mathcal{F}$.

Note that the unity function 1 does not belong to $\mathcal{F}$ or $\mathcal{L}$, but is automatically an element of $\mathcal{F}_\star$ and $1 \star f = f \star 1 = f, \forall f \in \mathcal{F}_\star$. So, $\mathcal{F}_\star$ is an involutive algebra with unity.

In the next section, we introduce a concrete example of a normal subalgebra $\mathcal{F} \subset \mathcal{L}$ and its extension $\mathcal{F}_*$ suitable for quantizing the phase space $T^*\mathcal{M}$. In the case $\mathcal{M} = \mathbb{R}^n$, similar extensions were used in [15, 61–64].

3. Quantization and dequantization over $T^*\mathcal{M}$

Let $\mathcal{M}$ be a smooth oriented manifold with an affine torsion-free connection Γ and a smooth positive measure $\mathrm{d}m$.

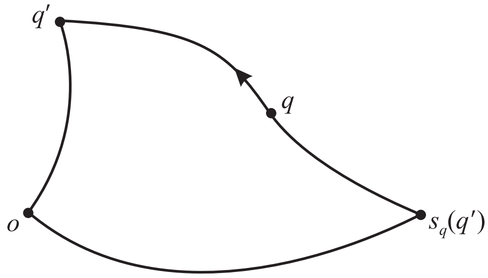

Figure 1. Geodesic triangle $\pi_q(q')$ in $\mathcal{M}$.

We assume that $\mathcal{M}$ is geodesically simply connected, that is, every pair of points is connected by a unique geodesic, and moreover this geodesic is infinitesimally isolated (has no conjugate points).

For any $q \in \mathcal{M}$, we use the notations

$$V_q = \underline{\exp}_q^{-1}, \qquad s_q = \underline{\exp}_q(-V_q),$$
$$j_q = 2^{-n}|\det(\partial V_q + \partial V_q(s_q))|, \qquad \partial V_q \equiv \frac{\partial V_q}{\partial q},$$
$$J_q = j_q \frac{\mathcal{D}m(s_q)}{\mathcal{D}m}, \qquad e_q \equiv \left.\frac{\mathcal{D}m(\underline{\exp}_q(v))}{\mu(q)\mathcal{D}v}\right|_{v=V_q}.$$

In these formulae, one has the following objects:

- the exponential map $\underline{\exp}_q : T_q\mathcal{M} \to \mathcal{M}$ which is everywhere non-degenerate;
- for any $q' \in \mathcal{M}$, the vector $V_q(q') \in T_q\mathcal{M}$ is the velocity on the geodesic connecting q with q' in unit time;
- the mapping $s_q : \mathcal{M} \to \mathcal{M}$ is the geodesic reflection about point q, $s_q{}^2 = \mathrm{i}d$;
- the Jacobian $j_q \in C^\infty(\mathcal{M})$ is invariant under the reflection s_q;
- $\mu > 0$ denotes the density of the measure on $\mathcal{M}$, so that $\mathrm{d}m(q) = \mu(q)\,\mathrm{d}q$;
- the Jacobian $\mathcal{D}m(s_q)/\mathcal{D}m \in C^\infty(\mathcal{M})$ is obtained by transforming the measure $\mathrm{d}m$ under the diffeomorphism s_q;
- the Jacobian $e_q \in C^\infty(\mathcal{M})$ determines the transformation of the measure $\mathrm{d}m$ under the diffeomorphism $\underline{\exp}_q$.

In addition, let F be a closed 2-form on $\mathcal{M}$. We fix an arbitrary point $o \in \mathcal{M}$ and define the function $\Phi_q \in C^\infty(\mathcal{M})$ as

$$\Phi_q(q') = \int_{\pi_q(q')} F. \tag{3.1}$$

Here, $\pi_q(q')$ is a two-dimensional surface in $\mathcal{M}$ whose oriented boundary is composed of three geodesics (figure 1): the geodesics from q' to o, from o to $s_q(q')$ and from $s_q(q')$ back to q'. The values of the function Φ_q are just the magnetic flux through the surfaces π_q.

Now we are ready to define the quantizer $\{\Delta_x | x \in T^*\mathcal{M}\}$. This family of operators is determined by its integral kernels $\Delta_x(a, b)$

$$(\Delta_x\psi)(a) = \int_{\mathcal{M}} \Delta_x(a, b)\psi(b)\,\mathrm{d}m(b), \qquad \psi \in \mathcal{D}(\mathcal{M}).$$

Here and below, we denote by $\mathcal{D}(\mathcal{N}) \equiv C_0^\infty(\mathcal{N})$ the space of all compactly supported C^∞-functions on a manifold $\mathcal{N}$.

Let $x = (q, p)$, so that $p \in T_q^*\mathcal{M}$. We set

$$\Delta_x(a,b) \stackrel{\text{def}}{=} \frac{1}{(\pi\hbar)^n}\sqrt{J_q(a)}\exp\left\{\frac{\mathrm{i}}{\hbar}(2pV_q(a)+\Phi_q(a))\right\}\delta_{s_q(a)}(b), \tag{3.2}$$

where δ is the delta function on $\mathcal{M}$ with respect to the measure dm; the product $pV_q(a)$ represents the natural pairing of the covector p and the tangent vector $V_q(a)$ in $T_q\mathcal{M}$.

Lemma 1. *The family of operators* $\{\Delta_x\}$ *defined by the integral kernels (3.2) obeys properties (2.6) and so is a quantizer:*

$$\begin{aligned}
&\overline{\Delta_x(a,b)} = \Delta_x(b,a),\\
&(2\pi\hbar)^n \int_{\mathcal{M}}\int_{\mathcal{M}} \Delta_x(a,b)\Delta_y(b,a)\,\mathrm{d}m(a)\,\mathrm{d}m(b) = \delta_x(y),\\
&(2\pi\hbar)^n \int_{T^*\mathcal{M}} \Delta_x(a,b)\Delta_x(c,d)\,\mathrm{d}x = \delta_a(d)\delta_b(c).
\end{aligned} \tag{3.3}$$

The proof follows directly from definition (3.2) by simple computation of integrals containing delta functions.

For some manifolds $\mathcal{M}$, the function j_q may be unbounded. In this case, the quantizer Δ_x is an unbounded operator but remains self-adjoint with domain $\mathcal{D}(\Delta_x) = \{\psi \in \mathcal{H} | \int j_q|\psi|^2\,\mathrm{d}m < \infty\}$. The family $\{\Delta_x\}$ and all x-derivatives are strongly continuous on $\mathcal{D}(\mathcal{M})$.

Recall that the operators $\hat{f}$ are defined by formula (2.4). Let $\mathfrak{F}$ denote the integral kernel of the operator $\hat{f}$,

$$(\hat{f}\psi)(a) = \int_{\mathcal{M}} \mathfrak{F}(a,b)\psi(b)\,\mathrm{d}m(b), \qquad \psi \in \mathcal{D}(\mathcal{M}). \tag{3.4}$$

For simplicity, we assume that $\mathfrak{F} \in \mathcal{D}(\mathcal{M}\times\mathcal{M})$.

Applying formula (2.5), one obtains the analogue of the Wigner transform.

Lemma 2. *The symbol f is constructed from the kernel $\mathfrak{F}$ via*

$$f(q,p) = \int_{T_q\mathcal{M}} \mathrm{e}^{-\mathrm{i}up/\hbar}(k\mathfrak{F})\left(\underline{\exp}_q\left(\frac{u}{2}\right), \underline{\exp}_q\left(-\frac{u}{2}\right)\right)\mu(q)\,\mathrm{d}u. \tag{3.5}$$

Here, the function k is given by

$$k(a,b) \stackrel{\text{def}}{=} (j_{a\vee b}(a)e_{a\vee b}(a)e_{a\vee b}(b))^{1/2}\exp\{-\mathrm{i}\Phi_{a\vee b}(a)/\hbar\}, \tag{3.6}$$

and $a \vee b$ denotes the geodesic midpoint (figure 2), that is

$$s_{a\vee b}(a) = b. \tag{3.7}$$

The inverse transform from symbol to kernel results from (2.4). Denote the Fourier image of f in the momentum variable by

$$f^{\sim}(q,u) \stackrel{\text{def}}{=} \frac{1}{(2\pi\hbar)^n}\int_{T_q^*\mathcal{M}} \mathrm{e}^{\mathrm{i}up/\hbar} f(q,p)\frac{\mathrm{d}p}{\mu(q)}. \tag{3.8}$$

Lemma 3. *The integral kernel $\mathfrak{F}$ of the operator corresponding to the symbol $f \in \mathcal{D}(T^*\mathcal{M})$ is given by*

$$\mathfrak{F}(a,b) = \frac{f^{\sim}(a\vee b, a\wedge b)}{k(a,b)}, \tag{3.9}$$

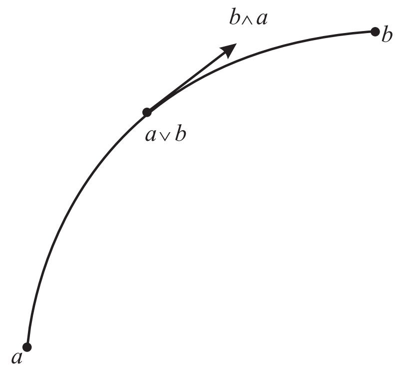

Figure 2. Geodesic midpoint and velocity in $\mathcal{M}$.

where $a \vee b$ is the geodesic midpoint (3.7) and $a \wedge b$ is the geodesic velocity at the midpoint (figure 2)

$$a \wedge b \stackrel{\text{def}}{=} V_{a\vee b}(a) - V_{a\vee b}(b) = 2V_{a\vee b}(a). \tag{3.10}$$

In the flat case $\mathcal{M} = \mathbb{R}^n$, one has $|k(a,b)| = 1$, and (3.5) and (3.9) become the standard Wigner transform in the presence of a magnetic field.

Using (3.9), one can readily compose the product of two operators and find a simple composition rule in terms of Fourier-imaged symbols. To formulate the result, let us recall that the tangent bundle $T\mathcal{M}$ is endowed with a natural groupoid multiplication [27]

$$n', n'' \mapsto n' \circ n'' \tag{3.11}$$

by means of the left and right (target and source) mappings

$$\tilde{l} : T\mathcal{M} \to \mathcal{M}, \qquad \tilde{l}(n) \stackrel{\text{def}}{=} \underline{\exp}_q\left(\frac{u}{2}\right),$$
$$\tilde{r} : T\mathcal{M} \to \mathcal{M}, \qquad \tilde{r}(n) \stackrel{\text{def}}{=} \underline{\exp}_q\left(-\frac{u}{2}\right), \qquad n \equiv (q,u) \in T\mathcal{M}.$$

Namely, the product (3.11) of two elements $n', n'' \in T\mathcal{M}$ is well determined iff $\tilde{r}(n') = \tilde{l}(n'')$, and in this case one has $\tilde{l}(n' \circ n'') = \tilde{l}(n'), \tilde{r}(n' \circ n'') = \tilde{r}(n'')$. Note that the mappings $\tilde{l}, \tilde{r}$ themselves can be expressed in terms of the groupoid multiplication as

$$\tilde{l}(n) = n \circ n^{-1}, \qquad \tilde{r}(n) = n^{-1} \circ n,$$

where $n^{-1} \equiv (q, -u)$ is the element inverse to n in $T\mathcal{M}$.

Lemma 4. *Composition of Fourier-imaged symbols over $T\mathcal{M}$ is given by the groupoid modified convolution*

$$(f^{\sim} \odot g^{\sim})(n) = \int_{n=n'\circ n''} \frac{\kappa(n)}{\kappa(n')\kappa(n'')} f^{\sim}(n')g^{\sim}(n'')\,\mathrm{d}m. \tag{3.12}$$

Here, n, n', n'' are points from $T\mathcal{M}$, the function $\kappa(n) \equiv k(\tilde{l}(n), \tilde{r}(n))$ is given by (3.6) and by the left and right mappings of the groupoid structure (3.11). The integration in (3.12) is taken with respect to the measure $\mathrm{d}m(\tilde{r}(n')) = \mathrm{d}m(\tilde{l}(n''))$ over the manifold $\mathcal{M}$.

Formula (3.12) belongs to the class of Connes' type tangential groupoid quantization formulae [4, 27, 31]. Note that in the convolution integrand (3.12), we have an additional *groupoid cocycle*

$$C(n', n'') = \frac{\kappa(n' \circ n'')}{\kappa(n')\kappa(n'')} = |C(n', n'')| \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{\Delta(n',n'')} F\right\}. \tag{3.13}$$

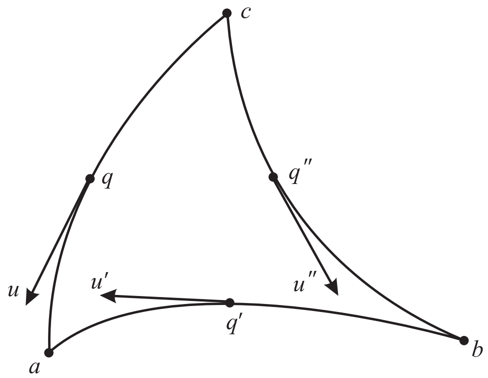

Figure 3. Geodesic triangle in $\Delta(n', n'')$ in $\mathcal{M}$.

The cocycle property

$$C(n, m \circ l)C(m, l) = C(n \circ m, l)C(n, m) \tag{3.14}$$

guarantees the associativity of the modified groupoid convolution (3.12).

The phase of the cocycle (3.13) is just the magnetic flux through the triangle $\Delta(n', n'')$ in $\mathcal{M}$ bounded by geodesics (figure 3) with midpoints q, q', q'' and mid-velocities u, u', u'' such that

$$n' = (q', u'), \qquad n'' = (q'', u''), \qquad n' \circ n'' = (q, u).$$

This phase is similar to its form in the Euclidean case [4], but now it also senses the non-Euclidean connection on $\mathcal{M}$. On the phase space level, the property (3.14) is equivalent to the Stokes theorem applied to the geodesic tetrahedron in $\mathcal{M}$ with sides corresponding to elements $n, m, l, m \circ l, n \circ m, l \circ n, n \circ m \circ l \in T\mathcal{M}$.

The amplitude $C(n', n'')$ of the cocycle (3.13) obtained from (3.6) is

$$|C(n', n'')| = \left(\frac{j_q(c)}{j_{q'}(a) j_{q''}(b)} \cdot \frac{e_q(a) e_q(c)}{e_{q'}(a) e_{q'}(b) e_{q''}(b) e_{q''}(c)} \right)^{1/2}.$$

This amplitude is an additional 'geodesic' contribution to the groupoid cocycle structure (3.13).

Obviously, one has the following proposition.

Proposition 1. *The quantization mapping $f \to \hat{f}$, defined by (3.4) and (3.9), is an isomorphism between the algebra $\mathcal{L} = L^2(T^*\mathcal{M}, \mathrm{d}l)$ and the algebra of all Hilbert–Schmidt operators acting on the Hilbert space $\mathcal{H} = L^2(\mathcal{M}, \mathrm{d}m)$. The quantum product $\star$ in the algebra $\mathcal{L}$ is generated by the groupoid modified convolution (3.12) as follows:*

$$(f \star g)^{\sim} = f^{\sim} \odot g^{\sim}. \tag{3.15}$$

Now we would like to extend the quantization mapping to a wider algebra.

Examining formula (3.12), one can easily see that the class $\mathcal{D}(T\mathcal{M})$ is invariant with respect to the $\odot$ convolution. Moreover, the class of C^∞-functions on TM with compact support in tangential directions is a $\mathcal{D}(T\mathcal{M})$-module with respect to this convolution. Thus, one is led to the following definitions.

Denote by $\mathcal{F} = \mathcal{F}(T^*\mathcal{M})$ the space of functions on $T^*\mathcal{M}$ whose momentum Fourier image belongs to $\mathcal{D}(T\mathcal{M})$.

Further, denote by $\mathcal{P} = \mathcal{P}(T^*\mathcal{M})$ the space of C^∞-functions on $T^*\mathcal{M}$ which are polynomials in momenta. Let $\mathcal{P}^l \subset \mathcal{P}$ be the subspace which consists of polynomials of degree l in momenta.

From the above discussion and the definitions of section 2, we have the following propositions.

Proposition 2. *$\mathcal{F}$ is a normal subalgebra of the Hilbert algebra $\mathcal{L} = L^2(T^*\mathcal{M})$.*

Proposition 3. *The space $\mathcal{P}$ is an involutive subalgebra in $\mathcal{F}_\star$ with the gradation $\mathcal{P}^l \star \mathcal{P}^m \subset \mathcal{P}^{l+m}$. The commutator in the algebra $\mathcal{P}$ is graded as follows:*

$$[\mathcal{P}^l, \mathcal{P}^m]_\star \subset \mathcal{P}^{l+m-1}, \qquad l, m \geqslant 0, \tag{3.16}$$

where $\mathcal{P}^{-1} \equiv 0$. In particular, $[\mathcal{P}^0, \mathcal{P}^0]_\star = 0$.

For any $f \in \mathcal{F}_\star$, the operator

$$\hat{f} : \mathcal{D}(\mathcal{M}) \to \mathcal{D}'(\mathcal{M}) \tag{3.17}$$

is defined by its bilinear form

$$(\hat{f}\psi, \chi)_\mathcal{H} = \langle f, \rho_{\psi|\chi}\rangle. \tag{3.18}$$

Here, $\rho_{\psi|\chi}$ is the *Wigner function*

$$\rho_{\psi|\chi}(x) \stackrel{\text{def}}{=} (\Delta_x\psi, \chi)_\mathcal{H}. \tag{3.19}$$

It is evident that $\rho_{\psi|\chi} \in \mathcal{F}$ if $\psi, \chi \in \mathcal{D}(\mathcal{M})$ and so the 'matrix elements' (3.18) are well defined. In the particular case where $f \in \mathcal{F}$, this definition of the operator $\hat{f}$ coincides with the above definitions (3.4), (3.9).

As one would expect, the Wigner function obeys the usual probability interpretation together with the associated bound condition

$$\int_{T_q^*\mathcal{M}} \rho_{\psi|\psi}(q, p)\,\mathrm{d}p = |\psi(q)|^2, \qquad |\rho_{\psi|\psi}(q, p)| \leqslant \frac{1}{(\pi\hbar)^n}\int_\mathcal{M} (j_q)^{1/2}|\psi|^2\,\mathrm{d}m.$$

Lemma 5. *If $f \in \mathcal{P}$, then $\hat{f}$ maps $\mathcal{D}(\mathcal{M})$ into $\mathcal{D}(\mathcal{M})$; moreover, it is a differential operator. If $f \in \mathcal{P}^l$, then $\hat{f}$ is a differential operator of order l.*

Let us consider now some basic examples of functions from $\mathcal{F}_\star$ or $\mathcal{P}$ and the operators (3.17) corresponding to them.

Example 1. Let 1 be the unit function on $T^*\mathcal{M}$. Then, $1 \in \mathcal{F}_\star$ and $\hat{1} = I$ is the identity operator in $L^2(\mathcal{M}, \mathrm{d}m)$.

Let δ_x be the delta function on $T^*\mathcal{M}$ concentrated at the point x. Then, $\delta_x \in \mathcal{F}_\star$. The corresponding operator is the quantizer (3.2),

$$\widehat{\delta_x} = \Delta_x.$$

In the Euclidean case, see [56, 57].

Example 2. Let $f \in \mathcal{P}^1$. Then, f is represented by the sum $f(q, p) = \varphi(q) + f_W(q, p)$ with $\varphi \in C^\infty(\mathcal{M})$, $f_W(q, p) = pW(q)$ and where W is a vector field on $\mathcal{M}$. Obviously, $\widehat{\varphi} = \varphi$ (the multiplication operator) and

$$\widehat{f_W} = -\mathrm{i}\hbar\left(W + \tfrac{1}{2}\,\mathrm{div}\,W\right) - A^o W. \tag{3.20}$$

Here, on the right-hand side, the field W is considered as a first-order differential operator and div W denotes the divergence of W with respect to the measure dm. The function $A^o W$ in (3.20) is the pairing of the vector field W with the 1-form A^o on $\mathcal{M}$ and A^o is a primitive of the Faraday 2-form, i.e.

$$\mathrm{d}A^o = F.$$

This 1-form is uniquely determined by the radial gauge condition that $A^o(q)$ is *perpendicular to the velocity of the geodesic connecting o with q*;

$$A^o(q)V_q(o) = 0. \tag{3.21}$$

An explicit formula for A^o is the following [65]:

$$A^o(q) = \int_o^q \left(\frac{\partial Q}{\partial q}\right)^* F(Q)\,\mathrm{d}Q, \tag{3.22}$$

where Q is the geodesic from o to q, and the integral (3.22) is taken along this geodesic. The magnetic potential (3.22) can be interpreted as the average of the Lorentz force $F(Q)\dot{Q}$ with respect to the Jacobi field along the geodesic.

At this stage, it is useful to discuss the relationship between various gauge choices and structure of the quantizer. Definition (3.2) of Δ_x makes no reference to any particular magnetic potential. However, the magnetic phase flux Φ_q, when written as a line integral, is expressed in terms of the radial gauge by

$$\Phi_q(a) = \int_{s_q(a)}^a A^o. \tag{3.23}$$

Here, the integral is taken along the geodesic on $\mathcal{M}$ connecting $b = s_q(a)$ to a. The ao and ob geodesic contributions to $\Phi_q(a)$ vanish by virtue of the radial gauge condition (3.21). Examples 2 and 3 demonstrate the dependence on just this special radial gauge 1-form A^o. If one modifies the definition of the quantizer by replacing A^o in (3.23) by a magnetic vector potential in some other gauge, say A, then the potential A will replace A^o in the quantization of the momentum coordinate, (3.20). We conclude that although the quantum algebra is gauge invariant, its representation depends on the gauge choice.

In the Euclidean case $\mathcal{M} = \mathbb{R}^n$, the 1-form (3.22) coincides with the Valatin potential [66] and condition (3.21) is the Dirac gauge condition (see details in [4]).

Example 3. Let M be a covariantly constant bivector field on $\mathcal{M}$ and $f_M(q,p) \equiv M^{jk}(q)p_j p_k, \forall p \in T_q^*\mathcal{M}$. Then, $f_M \in \mathcal{P}^2$ and

$$\widehat{f_M} = \left(-\mathrm{i}\hbar\underline{\nabla}_j - A_j^o\right)M^{jk}\left(-\mathrm{i}\hbar\underline{\nabla}_k - A_k^o\right),$$

where $\underline{\nabla}$ is the covariant derivative defined by the connection $\underline{\Gamma}$ on $\mathcal{M}$ and A_j^o are components of the 1-form (3.22).

In particular, let $\underline{\Gamma}$ be the Levi–Civita connection and $g = ((g_{jk}))$ be the metric tensor on $\mathcal{M}$. Then,

$$\widehat{g^{jk}(q)p_j p_k} = g^{jk}\left(-\mathrm{i}\hbar\underline{\nabla}_j - A_j^o\right)\left(-\mathrm{i}\hbar\underline{\nabla}_k - A_k^o\right).$$

If the magnetic field is absent, this reduces to $-\hbar^2\triangle$ (the Laplace operator on $\mathcal{M}$).

Example 4. Let W be a vector field on $\mathcal{M}$ and $r \in C^\infty(\mathcal{M})$. Consider the function on $T^*\mathcal{M}$ given by

$$r_W(q,p) \equiv r(q)\exp\left\{\frac{\mathrm{i}}{\hbar}pW(q)\right\}, \qquad p \in T^*{}_q\mathcal{M}. \tag{3.24}$$

Suppose the following condition holds:

$$\textit{the matrix } \tfrac{1}{2}\partial W(b)/\partial b[\partial V_a(b)/\partial b]^{-1} \textit{ has no eigenvalues } \pm 1 \textit{ for any } a, b \in \mathcal{M}. \tag{3.25}$$

Then, $r_W \in \mathcal{F}_\star$.

Indeed, by computing $r_W{}^\sim$ and transforming this to the kernel form (3.9), one obtains the delta function $\delta(W(a \vee b) + a \wedge b)$, which must be a well-defined distribution in both a and b (in order to achieve the embedding $r_W \in \mathcal{F}_\star$). Thus, it is sufficient to assume that both determinants

$$\det \frac{\partial}{\partial a}(W(a \vee b) + a \wedge b) \neq 0, \qquad \det \frac{\partial}{\partial b}(W(a \vee b) + a \wedge b) \neq 0 \tag{3.26}$$

are non-zero for all $a, b \in \mathcal{M}$. This is equivalent to condition (3.25).

The operator corresponding to the symbol r_W, subject to condition (3.25), acts in $\mathcal{H}$ as

$$(\widehat{r_W}\psi)(a) = \frac{J_q\big(\underline{\exp}_q\big(-\frac{1}{2}W(q)\big)\big)^{1/2}}{\big|\det\big(\partial V_q(a) + \frac{1}{2}\partial W(q)\big)\big|} r(q)\,\mathrm{e}^{-\mathrm{i}\Phi_q(a)/\hbar}\psi\left(\underline{\exp}_q\left(\frac{1}{2}W(q)\right)\right)\Bigg|_{q=Q(a)}, \tag{3.27}$$

where $Q(a)$ is the solution of the equation

$$\underline{\exp}_Q\big(-\tfrac{1}{2}W(Q)\big) = a. \tag{3.28}$$

Now, consider the specific class of *autoparallel vector fields*, i.e. those satisfying the identity

$$\underline{\nabla}_W W = 0 \qquad \text{or} \qquad W^j\partial_j W^k + \underline{\Gamma}^k_{js}W^jW^s = 0.$$

In this case, the flow $R^t : \mathcal{M} \to \mathcal{M}$ of the field W is given by

$$R^t(q) = \underline{\exp}_q(tW(q)). \tag{3.29}$$

Note that condition (3.25) is satisfied automatically and equation (3.28) now reads $R^{-1/2}(Q) = a$, and so, $Q(a) = R^{1/2}(a)$.

Moreover, $\underline{\exp}_q\big(\frac{1}{2}W(q)\big) = \exp_a(W(a))$ and so $\psi\big(\underline{\exp}_q\big(\frac{1}{2}W(q)\big)\big|_{q=Q(a)} = \psi(R^1(a))$.

Thus, the operator $\widehat{r_W}$ acts as a shift operator along the trajectories of the field W.

Lemma 6. *Let W be an autoparallel vector field on $\mathcal{M}$, generating the flow (3.29). Set*

$$r^t(q) \stackrel{\text{def}}{=} \det DV_q(R^{t/2}(q)) \left(\frac{\det \mathrm{d}R^{t/2}(q) \cdot \det \mathrm{d}R^{-t/2}(q)}{j_q(R^{t/2}(q))}\right)^{1/2}. \tag{3.30}$$

Then, the family

$$\widehat{r^t_W} \equiv \widehat{r^t \exp\left\{\frac{\mathrm{i}t}{\hbar} f_W\right\}} \qquad (t \in \mathbb{R}), \tag{3.31}$$

where $f_W(q, p) \equiv pW(q)$, forms a one-parameter group of unitary operators in $\mathcal{H}$ acting by the formula

$$\big(\widehat{r^t_W}\psi\big)(a) = \sqrt{\frac{\mathcal{D}m(R^t(a))}{\mathcal{D}m(a)}} \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{\pi^t(a)} F\right\}\psi(R^t(a)). \tag{3.32}$$

Here, the boundary of a surface $\pi^t(a) \subset \mathcal{M}$ is composed of three geodesics connecting the points $o \to R^t(a) \to a \to o$.

Indeed, the operator given by (3.32) is automatically unitary. So, to prove the lemma, we just need to compare formulae (3.32) and (3.27), and choose an appropriate amplitude function r.

Note that the mapping $R^t : \mathcal{M} \to \mathcal{M}$ can be naturally lifted up to the mapping

$$\gamma_W^t : T^*\mathcal{M} \to T^*\mathcal{M}, \qquad \gamma_W^t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} R^t(q) \\ \mathrm{d}R^t(q)^{-1*}(p + \beta^t) \end{pmatrix}. \tag{3.33}$$

Here, the covector $\beta^t \in T_q^*\mathcal{M}$ is defined by $\beta^t = A(q, R^t(q))$, where

$$A(q, a) \stackrel{\text{def}}{=} \int_a^q \left(\frac{\partial Q}{\partial q}\right)^* F(Q)\,\mathrm{d}Q. \tag{3.34}$$

The potential $A(q, a) \in T_q^*\mathcal{M}$ is the version of (3.22) with the initial point a instead of o: $d_q A(q, a) = F(q)$, $A(q, a)V_q(a) = 0$. In particular, if $a = o$, then $A(q, o) \equiv A^o(q)$ in the notation of (3.22).

Corollary 1. *Let W be an autoparallel vector field on $\mathcal{M}$ and $g \in \mathcal{P}^1$. Then, the following permutation formula holds on the dense domain $\mathcal{D}(\mathcal{M}) \subset \mathcal{H}$:*

$$\widehat{r_W^t} \cdot \widehat{g} \cdot \widehat{r_W^t}^{-1} = \widehat{g_W^t}, \qquad g_W^t \stackrel{\text{def}}{=} \gamma_W^{t\,*} g. \tag{3.35}$$

Thus, we see that the unitary operators $\widehat{r_W^t}$ with symbols of exponential type (3.31) play the role of the quantum transformations corresponding to the classical symplectic transformations γ_W^t (3.33). Permutation formula (3.35) belongs to the general class of Fock-type formulae [67] which relate classical symplectic transformations to quantum unitary operators (see also in [38, 45, 64]).

Properties (3.16) and (3.35) which we derived form the definition of the quantizer (3.2) actually determine the quantization uniquely.

Proposition 4. *If a quantization obeys the axioms (2.6) and the graded commutator property (3.16), and if for any autoparallel vector field W on $\mathcal{M}$ there is a function $r^t \in C^\infty(\mathcal{M})$ such that the operator $\widehat{r_W^t}$ (3.31) is unitary and the property (3.35) holds, then this quantization coincides with the one defined by (3.5) and (3.9).*

These conditions for uniqueness are, in a sense, analogous to those known in the Euclidean case with no magnetic field [39–41], but our proposition 4 uses different logical assumptions.

We call the quantization defined by (3.5) and (3.9) a *magneto-geodesic quantization*.

Observe that if the form of the quantizer kernel is *a priori* assumed to be of the type (3.2) having the phase and δ-function structure given there but with unknown amplitude, then requiring that Δ_x obey axioms (2.6) fixes the amplitude. The additional properties (3.16) and (3.35) in proposition 4 were introduced to uniquely select the δ-function and the phase function of (3.2).

Note that in the case $F = 0$ (no magnetic field), the quantization which we uniquely identify above, should coincide with the one suggested in [31]. Our formulae (3.5) and (3.9) in this case are similar to formulae (7) from [31], but a non-trivial recalculation of the Jacobian in the cochain (3.6) is required in order to bring it into the form used in [31].

Comparison of our quantization formulae (3.5) and (3.9) with the versions introduced in [19, 23, 25, 26, 28, 30, 33] shows a difference in the amplitude factor k of (3.6) and this corresponds with the fact that the last two axioms in (2.6) do not hold for those other quantizations.

4. Magneto-geodesic reflections

The magneto-geodesic quantization defined in the previous section is based on the quantizer structure. We now analyse this structure from the viewpoint of symplectic transformations in the phase space $\prec T^*\mathcal{M}, \omega\succ$.

First note that the quantizer Δ_x can be decomposed into the product of its unitary part $\overset{\circ}{\Delta}_x$ and its modulus $|\Delta_x|$ as follows:

$$\Delta_x = |\Delta_x| \cdot \overset{\circ}{\Delta}_x, \tag{4.1}$$

where $x = (q, p) \in T^*\mathcal{M}$. Here, $|\Delta_x|$ is the positive square root of $\Delta_x^\dagger \Delta_x = \Delta_x^2$. It has the form of a multiplication operator $|\Delta_x| = (\pi\hbar)^{-n}\sqrt{j}_q$. The unitary part $\overset{\circ}{\Delta}_x$ is

$$\overset{\circ}{\Delta}_x = \sqrt{\frac{\mathcal{D}m(s_q)}{\mathcal{D}m}} \exp\left\{\frac{2\mathrm{i}}{\hbar} pV_q + \frac{\mathrm{i}}{\hbar}\Phi_q\right\} s_q^*, \tag{4.2}$$

where s_q^* is the operator in $\mathcal{H} = L^2(\mathcal{M}, \mathrm{d}m)$ generated by the geodesic reflection

$$(s_q^*\psi)(q') = \psi(s_q(q')), \qquad \psi \in \mathcal{H}.$$

One can continue this decomposition and represent $\overset{\circ}{\Delta}_x$ as the product of two unitary factors

$$\overset{\circ}{\Delta}_x = E_x \cdot T_x, \tag{4.3}$$

which are defined by

$$E_x \equiv \exp\left\{\frac{2\mathrm{i}}{\hbar} pV_q + \frac{\mathrm{i}}{\hbar}\Phi_q\right\}, \qquad T_x \equiv \sqrt{\frac{\mathcal{D}m(s_q)}{\mathcal{D}m}} s_q^*. \tag{4.4}$$

Note that both E_x and T_x are unitary and they both commute with $|\Delta_x|$. In addition, T_x and $\overset{\circ}{\Delta}_x$ are self-adjoint.

Now we would like to associate the unitary factors in (4.3) with symplectic transformations of phase space. This is achieved by means of a Fock procedure like (3.35).

Proceed first with the operator T_x. For each $x = (q, p) \in T^*\mathcal{M}$, define the transformation t_x of the phase space by the following formula:

$$t_x \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} s_q(q') \\ \mathrm{d}s_q(q')^{-1*}\big(p' + \beta_q(q')\big) \end{pmatrix}. \tag{4.5}$$

Here, q' runs over $\mathcal{M}$, $p' \in T^*_{q'}\mathcal{M}$, and the 1-form β_q on $\mathcal{M}$ is determined by

$$\beta_q = A^o - s_q^* A^o, \tag{4.6}$$

where A^o is the 1-form (3.22) and s_q^* denotes the pullback of a 1-form. Obviously, the mapping t_x preserves the form ω (1.3).

Lemma 7. *For any $g \in \mathcal{P}^1$, the permutation formula*

$$T_x \hat{g} T_x^{-1} = \widehat{t_x^* g} \tag{4.7}$$

holds on the dense domain $\mathcal{D}(\mathcal{M}) \subset \mathcal{H}$.

Formula (4.7) is easily derived from (3.20) and (4.5). It relates the unitary operator T_x in the Hilbert space $\mathcal{H} = L^2(\mathcal{M}, \mathrm{d}m)$ to the symplectic transformation t_x on the phase space $\prec T^*\mathcal{M}, \omega\succ$.

Now we proceed in the same way with the operator E_x. For each $x = (q, p) \in T^*\mathcal{M}$, let us define the transformation e_x of phase space as follows:

$$e_x \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} q' \\ p' - 2\,\mathrm{d}V_q(q')^* p - \mathrm{d}\Phi_q(q') \end{pmatrix}. \tag{4.8}$$

Here, $\mathrm{d}V_q$ is the differential of the mapping $V_q = \underline{\exp}_q^{-1}$. This transformation preserves the form ω.

Lemma 8. *For any $g \in \mathcal{P}^1$, the permutation formula*

$$E_x \hat{g} E_x^{-1} = \widehat{e_x^* g} \tag{4.9}$$

holds on the dense domain $\mathcal{D}(\mathcal{M}) \subset \mathcal{H}$.

This formula relates the unitary E_x with the symplectic e_x and follows without difficulty from (3.20) and (4.4).

As a consequence of these calculations, we have

$$\overset{\circ}{\Delta}_x\ \hat{g}\overset{\circ}{\Delta}{}_x^{-1} = E_x T_x \hat{g} T_x^{-1} E_x^{-1} = \widehat{e_x^* t_x^* g},$$

for any $g \in \mathcal{P}^1$. Thus, one obtains the composition of two symplectic transformations

$$\sigma_x \overset{\mathrm{def}}{=} t_x \circ e_x. \tag{4.10}$$

Corollary 2. *The symplectic transformations $\{\sigma_x\}$ are related to the unitary part of the quantizer by the identity*

$$\overset{\circ}{\Delta}_x\ \widehat{g}\ \overset{\circ}{\Delta}{}_x^{-1} = \widehat{\sigma_x^* g}, \tag{4.11}$$

for any $g \in \mathcal{P}^1$ and any $x \in T^*\mathcal{M}$. These identities hold on the dense domain $\mathcal{D}(\mathcal{M}) \subset \mathcal{H}$.

From formulae (4.5) and (4.8) and from the equalities

$$\mathrm{d}\Phi_q(q') = \beta_q(q') - \alpha_q(q'), \qquad \alpha_q(q') \overset{\mathrm{def}}{=} A(q', q) - \mathrm{d}s_q(q')^* A(s_q(q'), q), \tag{4.12}$$

the symplectic mapping σ_x is determined to be

$$\sigma_x \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} s_q(q') \\ \mathrm{d}s_q(q')^{-1*}[p' - 2\,\mathrm{d}V_q(q')^* p + \alpha_q(q')] \end{pmatrix}. \tag{4.13}$$

Here, the potential A is given by (3.34). Note that the potential A and thereby the right-hand side above is gauge independent, although (4.5) and (4.8) are gauge dependent (they depend on a choice of the point o in the potential A^o in (3.22)).

Let us check the simplest properties of the family of mappings (4.13). If the running point $x' = (q', p')$ coincides with x, then $\sigma_x(x')|_{x'=x} = x$. Thus, the point x is a fixed point of the transformation σ_x.

Furthermore, from the evident quantum permutation relations

$$T_x E_x = E_x^{-1} T_x, \qquad T_x^2 = I,$$

one obtains the corresponding classical counterparts

$$e_x \circ t_x = t_x \circ e_x^{-1}, \qquad t_x^2 = \mathrm{id}.$$

Definition (4.10) then implies

$$\sigma_x^2 = t_x \circ e_x \circ t_x \circ e_x = t_x \circ t_x \circ e_x^{-1} \circ e_x = \mathrm{id}.$$

Thus, corollary 2 can be completed as follows.

Corollary 3. *The family $\{\sigma_x | x \in T^*\mathcal{M}\}$ of symplectic transformations on the space $\prec T^*\mathcal{M}, \omega\succ$ is given by formula (4.13) and possesses the properties:*

- *x is a unique and isolated fixed point of σ_x;*
- *each σ_x is a reflection, i.e. $\sigma_x^2 = \mathrm{id}$.*

This family of reflections can be considered as a lift to $T^*\mathcal{M}$ of the family of reflections $\{s_x\}$ given on $\mathcal{M}$. We call σ_x a *magneto-geodesic reflection*. The maps $\{\sigma_x\}$ generalize the family of magnetic reflections found in [5] for the phase space $\prec T^*\mathbb{R}^n, \omega\succ$ with the Euclidean connection on $\mathcal{M} = \mathbb{R}^n$. Now our reflections (4.13) combine both a non-trivial magnetic field and a non-trivial affine connection on $\mathcal{M}$.

We note that the correspondence between the quantizer and the phase space reflections was observed in [56] for the Euclidean space $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ with no magnetic field, see also [57, 58]. In this latter case, the reflections are just $\sigma_x(x') = 2x - x'$.

In the general phase space $T^*\mathcal{M}$, by using the magneto-geodesic reflections σ_x, one can easily define the notions of σ-midpoints and σ-reflective curves.

Namely, the point $x \in T^*\mathcal{M}$ is called a σ-midpoint between $x', x'' \in T^*\mathcal{M}$ if $x'' = \sigma_x(x')$. From (4.13), it follows that any two points from $T^*\mathcal{M}$ have a unique σ-midpoint.

A continuous curve passing through a point $x \in T^*\mathcal{M}$ is called *σ-reflective with respect to x* if it consists of pairs of points with the midpoint x. The projection of the σ-reflective curve from $T^*\mathcal{M}$ onto $\mathcal{M}$ is a reflective curve in $\mathcal{M}$, and vice versa, any reflective curve from $\mathcal{M}$ (in particular, the geodesic) can be lifted to a σ-reflective curve on $T^*\mathcal{M}$ (but of course, not uniquely).

5. Integral formula for quantum product

Next, we present an explicit formula for the product $f \star g$, expressed directly in terms of the functions f and g. The definition of $\star$-product was given in (3.15). In principle, one could use (3.15) to compute $f \star g$. But it is convenient to use the equivalent representation (2.7).

First we compute the distribution (2.8), that is the trace of the composition of three quantizers $\Delta_x\Delta_y\Delta_z$. Label the phase space points here by

$$\begin{aligned} &x = (a, \xi), \quad y = (b, \eta), \quad z = (c, \zeta), \qquad \text{where} \\ &\xi \in T_a^*\mathcal{M}, \quad \eta \in T_b^*\mathcal{M}, \quad \zeta \in T_c^*\mathcal{M}. \end{aligned} \tag{5.1}$$

Let (q, q') denote the two arguments of the integral kernel of the composition $\Delta_x\Delta_y\Delta_z$. Employing (3.2) gives

$$(2\pi\hbar)^n \cdot \mathrm{kernel}(\Delta_x\Delta_y\Delta_z)(q, q') = T^{\xi,\eta,\zeta}_{a,b,c}(q)\delta_{s_c s_b s_a(q)}(q'), \tag{5.2}$$

$$T^{\xi,\eta,\zeta}_{a,b,c}(q) = (\pi\hbar)^{-2n} \exp\big\{S^{\xi,\eta,\zeta}_{a,b,c}\big\}\varphi_{a,b,c}(q). \tag{5.3}$$

The phase and amplitude functions in (5.3) are

$$S^{\xi,\eta,\zeta}_{a,b,c}(q) = 2[\xi V_a(q) + \eta V_b(s_a(q)) + \zeta V_c(s_b s_a(q))] + \Phi_a(q) + \Phi_b(s_a(q)) + \Phi_c(s_b s_a(q)), \tag{5.4}$$

$$\varphi_{a,b,c}(q) = 2^n[J_a(q)J_b(s_a(q))J_c(s_b s_a(q))]^{1/2}. \tag{5.5}$$

Clearly, $T^{\xi,\eta,\zeta}_{a,b,c}(q)$ is a non-singular, continuous function of all its arguments.

The integral kernel (5.2) is singular. Therefore, in order to evaluate the trace of the corresponding operator, we first contract the distribution (5.2) with a test function $\phi \in \mathcal{D}(\mathcal{M})$ by the parameter $c \in \mathcal{M}$. This computation is implemented by using the formula

$$\int_{\mathcal{M}} \delta_{s_c s_b s_a(q)}(q')\phi(c)\,\mathrm{d}m(c) = \phi(c)\left|\frac{\mathcal{D}m(s_c s_b s_a(q))}{\mathcal{D}m(c)}\right|^{-1}_{c=q'\vee s_b s_a(q)}. \tag{5.6}$$

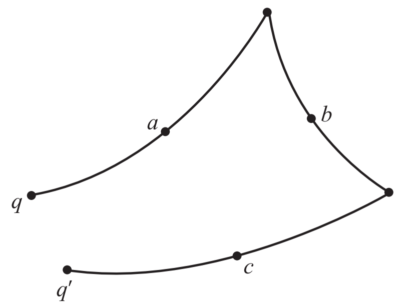

Figure 4. Mapping $s_c s_b s_a(q) = q'$.

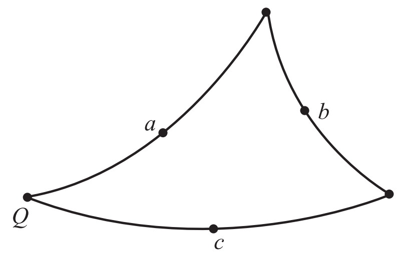

Figure 5. Fixed point Q of the compound reflection $s_c s_b s_a$.

On the right-hand side of this formula, the point c is taken to be the midpoint $q' \vee s_b s_a(q)$ (see figure 4).

In order to evaluate the trace, we have to put $q = q'$ and integrate over all $q \in \mathcal{M}$. Thus, from (5.2) and (5.6), one obtains

$$(2\pi\hbar)^n \int_{\mathcal{M}} \operatorname{Tr}\left(\Delta_{(a,\xi)}\Delta_{(b,\eta)}\Delta_{(c,\zeta)}\right)\phi(c)\,\mathrm{d}m(c) = \int_{\mathcal{M}} T^{\xi,\eta,\zeta}_{a,b,c}(q)\frac{\phi(c)}{\left|\frac{\mathcal{D}m(s_c s_b s_a(q))}{\mathcal{D}m(c)}\right|}\Bigg|_{c=q\vee s_b s_a(q)}\mathrm{d}m(q). \tag{5.7}$$

The mapping

$$q \mapsto c = q \vee s_b s_a(q) \tag{5.8}$$

is smooth, but it can be degenerate. The Jacobian of this mapping is

$$\left|\frac{\mathcal{D}m(c)}{\mathcal{D}m(q)}\right| = |\det(I - d(s_c s_b s_a)(q))| \cdot \left|\frac{\mathcal{D}m(s_c s_b s_a(q))}{\mathcal{D}m(c)}\right|^{-1}. \tag{5.9}$$

So, the degeneracy of (5.8) is controlled by the determinant

$$\mathcal{J}_{a,b,c}(q) \stackrel{\text{def}}{=} |\det(I - d(s_c s_b s_a)(q))|. \tag{5.10}$$

Here, the point c is assumed to be the image point of the mapping (5.8).

In each connected domain $\mathcal{M}_0 \subset \mathcal{M}$, where the Jacobian (5.10) is not zero, one can invert the mapping (5.8) and uniquely express q as a function of c (and of a, b as well): $q = Q(a, b, c)$. Obviously, Q is the fixed point of the composition of three reflections (see figure 5):

$$s_c s_b s_a(Q) = Q. \tag{5.11}$$

Inside the domain $\mathcal{M}_0$, the solution of this fixed-point problem exists and is unique. At the boundary of the domain $\mathcal{M}_0$, where the Jacobian (5.10) becomes zero, the solution of (5.11) is not infinitesimally isolated.

From (5.7) and (5.9), it follows that

$$(2\pi\hbar)^n \int_{\mathcal{M}} \mathrm{Tr}(\Delta_{(a,\xi)}\Delta_{(b,\eta)}\Delta_{(c,\zeta)})\phi(c)\,\mathrm{d}m(c) = \int_{\mathcal{M}} T^{\xi,\eta,\zeta}_{a,b,c}(q)\mathcal{J}_{a,b,c}(q)^{-1}\phi(c)\left|\frac{\mathcal{D}m(c)}{\mathcal{D}(q)}\right|\mathrm{d}m(q),$$

and so, one has the following formula for the distribution $K_\star$ in (2.8):

$$K_\star(x, y, z) = (2\pi\hbar)^n\,\mathrm{Tr}(\Delta_{(a,\xi)}\Delta_{(b,\eta)}\Delta_{(c,\zeta)}) = \sum_Q \frac{1}{\mathcal{J}_{a,b,c}(Q)} T^{\xi,\eta,\zeta}_{a,b,c}(Q). \tag{5.12}$$

The summation is taken over all fixed points of (5.11).

If the triple (a, b, c) is such that there is no solution of the fixed-point problem (5.11), then the value of the trace (5.12) is just zero.

If the triple (a, b, c) is such that there is a solution of the fixed-point problem (5.11) which is not infinitesimally isolated (the Jacobian (5.10) is zero), then the value of the trace (5.12) is infinite. The precise description of what this 'infinity' actually is, is given by the integral (5.7) (where there are no singularities at all).

Note that the amplitude factor on the right-hand side of formula (5.12) is derived from (5.3) and (5.5) and the definition of the Jacobian J_a at the beginning of section 3. Thus, we obtain the amplitude

$$\varphi(a, b, c) \stackrel{\text{def}}{=} \varphi_{a,b,c}(Q) = 2^n[j_a(Q)j_b(s_a(Q))j_c(s_b s_a(Q)) \cdot |\det d(s_c s_b s_a)(Q)|]^{1/2}, \tag{5.13}$$

where $Q = Q(a, b, c)$ is the solution of (5.11).

The phase of the exponential factor in (5.12) can be represented in the following geometrical form.

Lemma 9.

$$S^{\xi,\eta,\zeta}_{a,b,c}(Q) = \int_{\Sigma(x,y,z)} \omega. \tag{5.14}$$

Here, the symplectic form ω is determined by (1.3), the points $x, y, z \in T^*\mathcal{M}$ are given by (5.1) and $\Sigma(x, y, z)$ is a triangle in $T^*\mathcal{M}$ whose sides are σ-reflective curves with midpoints x, y, z.

Indeed, the symplectic area on the right-hand side of (5.14), by Stokes theorem, can be represented as the sum of three integrals

$$\int_{\Sigma(x,y,z)} \omega = \int_{Q'}^{Q} (\tilde p\,\mathrm{d}\tilde q + A^0) + \int_{Q''}^{Q'} (\tilde p\,\mathrm{d}\tilde q + A^0) + \int_{Q}^{Q''} (\tilde p\,\mathrm{d}\tilde q + A^0). \tag{5.15}$$

Here, $Q' = s_a(Q)$, $Q'' = s_b s_a(Q)$, each integral (5.15) is taken along the geodesics through the midpoints a, b, c, respectively, and $\tilde p \in T^*\mathcal{M}$ denotes the value of the momentum on the σ-reflective curves over these geodesics. The magnetic potential A^0 is a primitive of the Faraday form $\mathrm{d}A^0 = F$. Using (3.23), we conclude that the three integrals of A^0 in (5.15) correspond to three summands with the functions Φ_a, Φ_b, Φ_c in (5.4), and the three other integrals of $\tilde p\,\mathrm{d}\tilde q$ contribute to the terms containing the momenta ξ, η, ζ. For example, in the first integral of (5.15), one has

$$\int_{Q'}^{Q} \tilde p\,\mathrm{d}\tilde q = \int_{Q'}^{Q} (\mathrm{d}V_a(\tilde q)^*\xi - b_a(\tilde q))\,\mathrm{d}\tilde q = \int_{Q'}^{Q} \mathrm{d}(\xi V_a(\tilde q)) = \xi V_a(Q) - \xi V_a(Q') = 2\xi V_a(Q).$$

In the above formula, $b_a \equiv A^0 - \frac{1}{2}\,\mathrm{d}\Phi_a$ is used to make the curve $\{(\tilde q, \tilde p)\}$ to be σ-reflective with respect to the midpoint $x = (a, \xi)$.

Thus, from (2.7), (2.8) and formulae (5.12)–(5.14), we obtain the following result:

$$(f \star g)(x) = \frac{1}{(\pi\hbar)^{2n}} \iint_{C(\underline{x})} \sum \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{\Sigma(x,y,z)} \omega\right\} \frac{\varphi(\underline{x}, \underline{y}, \underline{z})}{\mathcal{J}(\underline{x}, \underline{y}, \underline{z})} f(y)g(z)\,\mathrm{d}y\,\mathrm{d}z. \tag{5.16}$$

In this formula,

- all objects are independent of the choice of the measure on the manifold $\mathcal{M}$ and are determined by an affine structure Γ and a closed 2-form F on $\mathcal{M}$;
- points x, y, z belong to the phase space $T^*\mathcal{M}$ and points $\underline{x}, \underline{y}, \underline{z}$ are their projections onto $\mathcal{M}$;
- the positive function $\varphi \in C^\infty(\mathcal{M} \times \mathcal{M} \times \mathcal{M})$ is given by (5.13);
- the Jacobian $\mathcal{J}(\underline{x}, \underline{y}, \underline{z}) = |\det(I - d(s_{\underline{z}} s_{\underline{y}} s_{\underline{x}})(Q))|$ is determined by the differential of the composition $s_{\underline{z}} s_{\underline{y}} s_{\underline{x}}$ of three Γ-geodesic reflections in $\mathcal{M}$ with respect to midpoints $\underline{z}, \underline{y}, \underline{x}$, where Q is the fixed point of this composition;
- the magnetic form ω (1.3) is integrated in (5.16) over surfaces (or membranes) which are 'triangles' $\Sigma(x, y, z)$ in $T^*\mathcal{M}$ whose sides are σ-reflective curves with midpoints x, y, z. The projection of $\Sigma(x, y, z)$ onto $\mathcal{M}$ are geodesic triangles $\underline{\Sigma}(\underline{x}, \underline{y}, \underline{z})$ with midpoints $\underline{x}, \underline{y}, \underline{z}$;
- the domain of integration in (5.16) is a cotangent 'cylinder' $C(\underline{x}) = T^*(\mathcal{M} \times_{\underline{x}} \mathcal{M})$ over the subset $\mathcal{M} \times_{\underline{x}} \mathcal{M}$ consisting of all those pairs of points $\underline{y}, \underline{z} \in \mathcal{M} \times \mathcal{M}$ for which the triangle $\underline{\Sigma}(\underline{x}, \underline{y}, \underline{z})$ exists. The sum $\sum$ is taken over all such triangles.

Theorem 1. *The associative product of functions over the phase space $T^*\mathcal{M}$, which corresponds via (2.3) to the magneto-geodesic quantization, is determined by formula (5.16). This formula acts directly on the subalgebra $\mathcal{F}(T^*\mathcal{M})$ of functions whose momentum Fourier image belongs to $\mathcal{D}(T\mathcal{M})$ and is extended to the algebra $\mathcal{F}_\star$ (and to its subalgebra $\mathcal{P}$) by the procedure (2.11).*

The subset $\mathcal{M} \times_{\underline{x}} \mathcal{M}$ can be called the *domain of influence* of the point $\underline{x} \in \mathcal{M}$. Inside $\mathcal{M} \times_{\underline{x}} \mathcal{M}$, the Jacobian $\mathcal{J}$ is not zero. In general, the domain of influence $\mathcal{M} \times_{\underline{x}} \mathcal{M}$ does not coincide with the whole $\mathcal{M} \times \mathcal{M}$. So, if the function $f \otimes g$ is localized outside of $C(\underline{x})$, then the integral (5.16) vanishes in a neighbourhood of the fibre $T^*_{\underline{x}}\mathcal{M}$.

As an example consider the Lobachevski plane, $\mathcal{M} = H^2$, given by the hyperboloid in three-dimensional Euclidean space: $H^2 \equiv \left\{q \in \mathbb{R}^3 \middle| q_1^2 + q_2^2 - q_3^2 = -1\right\}$. The Riemannian structure on H^2 is induced from the Euclidean structure on $\mathbb{R}^3$; the connection $\underline{\nabla}$ on H^2 is the usual Levi–Civita connection. In this case, it is known (cf [68, 69]) that the geodesic triangle with midpoints $\underline{x}\underline{y}\underline{z}$ exists iff

$$-1 < \det|\underline{x}\underline{y}\underline{z}| < 1 \tag{5.17}$$

(here $\underline{x}\underline{y}\underline{z}$ are considered to be 3-vectors and $|\underline{x}\underline{y}\underline{z}|$ denotes the 3×3 matrix of their components). Under condition (5.17), such a triangle is unique. If one fixes $\underline{y}, \underline{z}$, then the subset $H_{\underline{y}\underline{z}}$ of points $\underline{x} \in H^2$ obeying (5.17) looks like a tubular neighbourhood of the geodesic passing through b and c. Certainly, $H_{\underline{y}\underline{z}}$ is a proper subset in H^2, that is $H_{\underline{y}\underline{z}} \neq H^2$ if $\underline{y} \neq \underline{z}$. Thus, if $\underline{x} \notin H_{\underline{y}\underline{z}}$, then $(\underline{y}, \underline{z}) \notin H^2 \times_{\underline{x}} H^2$. Therefore, the domain of influence $H^2 \times_{\underline{x}} H^2$ is a proper subset in $H^2 \times H^2$.

We see that the quantum product (5.16) determines the distribution

$$K_\star(f \otimes g \otimes l) = \langle f \star g, l\rangle, \tag{5.18}$$

whose support, in general, does not coincide with the whole $T^*\mathcal{M} \times T^*\mathcal{M} \times T^*\mathcal{M}$. The topological boundary of this support can be considered as a *quantum front*

$$\mathrm{front}(\star) \stackrel{\mathrm{def}}{=} \partial(\mathrm{supp}\, K_\star). \tag{5.19}$$

From one side of this boundary, the distribution $K_\star$ is identically zero. We call this phenomenon a *front effect*.

Something close to this was mentioned in the interesting note [69] following ideas of geometric quantization, but no actual construction of any associative product was produced there.

The phase space of the type $T^*\mathcal{M}$, which we investigate in this paper, and in particular the space T^*H^2 seems to be the first instance where the front effect for the $\star$-product is mathematically identified.

The kernel $K_\star$ can be considered as a product of two δ-functions

$$K_\star(x, y, z) = (\delta_y \star \delta_z)(x).$$

The set $\mathrm{supp}(\delta_y \star \delta_z)$ is the cylinder $T^*\mathcal{M}_{\underline{yz}}$ over the domain $\mathcal{M}_{\underline{yz}} \subset \mathcal{M} \times \mathcal{M}$ (consisting of those $\underline{x}$ for which the geodesic triangle $\underline{\Sigma}(\underline{x}, \underline{y}, \underline{z})$ exists). Outside this cylinder, the distribution $\delta_y \star \delta_z$ is identically zero. The boundary of $T^*\mathcal{M}_{\underline{yz}}$ is a front of the 'wave' $\delta_y \star \delta_z$ which travels in the phase space $T^*\mathcal{M}$ when the points $\underline{y}$ and $\underline{z}$ move away from each other.

Note that it is not possible to detect the front effect in classical mechanics or even in the formal deformation approach where instead of exact associative product like (5.16), one uses a formal asymptotic power series in $\hbar$.

Also note that in the approach based on ideas of pseudodifferential operator theory [26, 28, 29, 34], where the $\star$-product is considered only on symbols whose p-Fourier transform is localized near zero, one can introduce under the integral (5.16) a cut-off function. This function is identically 1 for x, y, z close enough to each other, but becomes 0 if x, y, z move away from each other. In this way, one can eliminate all the difficulties related to the possible existence of conjugate points on geodesics or the non-existence of triangles. Formula (5.16), with the cut-off function, works for arbitrary affine manifolds $\mathcal{M}$; no front effect exists in this approach and no summation over multiple triangles is needed. Of course, this 'cut-off method', in general, destroys the associativity of the $\star$-product. But with this approach, one can keep associativity in the algebra $\mathcal{P}$ of symbols polynomial in momenta. Thus, we conclude that *formula (5.16) works in the algebra $\mathcal{P}$ over an arbitrary manifold $\mathcal{M}$*.

Looking forward to section 6, we note that the coefficients of the asymptotic expansion (1.6), (6.1) of the $\star$-products are insensitive to the cut-off function, since they are derived from the diagonal of the exact product (5.16). Thus, the $\hbar \to 0$ asymptotic expansion of the product (5.16) works over an arbitrary manifold $\mathcal{M}$.

We stress that the core result of theorem 1 is the explicit formula for an associative quantum product over the phase space $T^*\mathcal{M}$. This formula is exact, not a deformation one, and even not a semiclassical one. Under the semiclassical approach, the leading term of the asymptotics of the product kernel $K_\star$ is known over general phase spaces [70]. The membrane formula like (5.14) for the phase of $K_\star$ was first suggested in [71] as a formula for the 'action' on the graph of groupoid multiplication, and it was proved in [70], in the general symplectic case, that indeed (5.14) is the correct solution of the Cauchy problem for the phase function. In the Euclidean case, membrane formulae of a similar type were discovered by Berry [72] for the asymptotics of the Wigner function (see also [73] for solutions of the Cauchy problem). The magnetic version of these formulae was first obtained and investigated in detail in [4, 5], the case of symmetric spaces was studied in [74], and for general manifolds, see [75, 76].

6. Magneto-geodesic connection and $\hbar$-expansion of the quantum product

In this section, we investigate the asymptotic properties of the product $f \star g$ in the classical approximation as $\hbar \to 0$ and express the results in terms of magneto-geodesic covariant derivatives over phase space $T^*\mathcal{M}$. It is assumed that the functions f, g are $\hbar$ independent.

The integral (5.16) determining the quantum product contains a rapidly oscillating exponential factor and a smooth (non-oscillating) amplitude. The exponent phase has a stationary point at $y = x, z = x$, which is isolated and non-degenerate. Therefore, one can apply the standard stationary phase method [77] to derive the asymptotic expansion of the integral (5.16) as $\hbar \to 0$. The structure of this expansion is

$$(f \star g)(x) = \sum_{k \geqslant 0} \frac{1}{k!} \left(\frac{-\mathrm{i}\hbar}{2} \right)^k G_k(f \otimes g)(x). \tag{6.1}$$

Here, G_k are differential operators of order $2k$ acting on the function $f(y)g(z)$ and then restricted to the diagonal (the stationary point) $y = x, z = x$.

Since we know that the unity function 1 is the unit element for the $\star$-product, then

$$\begin{aligned} &G_0(f \otimes 1) = f, \\ &G_k(f \otimes 1) = G_k(1 \otimes f) = 0, \qquad k \geqslant 1. \end{aligned} \tag{6.2}$$

From (6.2), it follows, for instance, that

$$G_0(f \otimes g) = fg, \qquad G_1(f \otimes g) = \langle \mathrm{d}f, \Psi\, \mathrm{d}g \rangle,$$

where Ψ is a 2-tensor. From the property (2.9) and the involution property $\overline{f \star g} = \overline{g} \star \overline{f}$, we see that Ψ must be skew symmetric and real. In addition, the associativity of the $\star$-product implies the Jacobi identity for Ψ. Thus, Ψ is a Poisson tensor on $T^*\mathcal{M}$. It is easy to check that $\Psi = \Omega^{-1}$, where Ω is the matrix of the symplectic form ω in the exponent of (5.16).

Thus, the expansion of (6.1) takes the form

$$f \star g = fg - \frac{\mathrm{i}\hbar}{2}\{f, g\} - \frac{\hbar^2}{8} G_2(f \otimes g)\Big|_{\mathrm{diag}} + O(\hbar^3), \tag{6.3}$$

with the Poisson bracket $\{f, g\} \equiv \langle \mathrm{d}f, \Psi\, \mathrm{d}g \rangle$.

In the general theory of deformation quantization [42, 47, 48, 50], it was observed that the operator G_2 in expansions of star products like (6.3) has to be related to a certain phase space connection ∇, and moreover, G_2 can be written in the form

$$G_2 = (\nabla' \Psi \nabla'')^2 + \nabla' \mathcal{B} \nabla''. \tag{6.4}$$

Here, $\mathcal{B}$ is a certain 2-tensor and the primes mark the argument (first or second) of the function $f \otimes g$ to which the covariant derivative ∇ is applied.

For our specific (and exact) quantum product $\star$ over $T^*\mathcal{M}$, the generic form (6.4) can be verified and explicit formulae for ∇ and $\mathcal{B}$ can be found directly from the asymptotic expansion of the integral (5.16).

But we also could match the quantization with a phase space connection in another way. The quantization is given by the quantizer Δ_x, which in turn defines a family of symplectic transformations σ_x via (4.13). The transformations σ_x acting on $T^*\mathcal{M}$ are the phase space analogues of the geodesic reflections s_q acting on $\mathcal{M}$. These latter reflections are generated by the connection $\underline{\nabla}$ over $\mathcal{M}$. The Christoffel symbols of this connection are determined via s_q as follows:

$$\underline{\Gamma}(q) = -\tfrac{1}{2} D^2 s_q(q')\big|_{q'=q},$$

where D denotes derivatives with respect to argument $q' \in \mathcal{M}$. We can just mimic this formula to define a connection over $T^*\mathcal{M}$

$$\Gamma(x) = -\tfrac{1}{2} D^2 \sigma_x(x')\big|_{x'=x}, \tag{6.5}$$

where D denotes derivatives by the argument $x' \in T^*\mathcal{M}$. This formula indeed generates a connection for any family of reflections, and such a connection is automatically symplectic if these reflections are symplectic [70]. In our magneto-geodesic situation, we have all these properties of the family $\{\sigma_x\}$ (corollary 3; for the Euclidean case $\mathcal{M} = R^n$, see more details in [5]).

Note that the restriction of σ_x to the configuration space $\mathcal{M}$ coincides with s_q (see (4.13)), and so, the set of Christoffel symbols Γ (6.5) contains the Christoffel symbols $\underline{\Gamma}$ inside itself. Thus, Γ is an extension of $\underline{\Gamma}$ from the configuration space to the phase space.

Explicit calculation of the second derivatives in (6.5) using (4.13) yields

$$\Gamma^q_{qq} = \underline{\Gamma}(q), \qquad \Gamma^p_{qp} = \Gamma^p_{pq} = -\underline{\Gamma}(q), \qquad \Gamma^p_{qq} = pB(q) + C(q), \tag{6.5a}$$

where

$$B^m_{jkl} \overset{\text{def}}{=} \frac{1}{3} \underset{jkl}{\mathfrak{S}} \left(2\underline{\Gamma}^m_{js}\underline{\Gamma}^s_{kl} - \partial_j \underline{\Gamma}^m_{kl}\right), \qquad C_{jkl} \overset{\text{def}}{=} \frac{1}{3}(\underline{\nabla}_k F_{jl} + \underline{\nabla}_l F_{jk}), \tag{6.5b}$$

and the notation $\mathfrak{S}$ denotes cyclic summation. All other components of Γ vanish identically: $\Gamma^q_{pq} = \Gamma^q_{qp} = \Gamma^q_{pp} = \Gamma^p_{pp} = 0$.

Let ∇ be the covariant derivative over $T^*\mathcal{M}$ defined by Christoffel symbols (6.5a). We call it the *magneto-geodesic connection*. We stress again that this connection is symplectic with respect to the 'magnetic' symplectic structure ω (1.3), i.e.

$$\nabla\omega = 0.$$

We see that such a ∇ is certainly related to the quantizer and so also to the $\star$-product (5.16). We claim that this is actually the same connection which appears in (6.4) under the $\hbar$-expansion of integral (5.16).

Now we demonstrate how to prove this claim. Let us fix $x = (a, \xi) \in T^*\mathcal{M}$. Formula (5.4) implies that there is just one stationary point of the phase function $S = S^{\xi,\eta,\zeta}_{a,b,c}$ with respect to variables $y = (b, \eta)$ and $z = (c, \zeta)$, namely, this is the point $y = x, z = x$. Near this point only one summand in (5.16) contributes to the asymptotic expansion up to $O(\hbar^\infty)$, namely, this is the summand corresponding to small triangles $\Sigma = \Sigma(x, y, z)$ near the point x. Consider x as the origin of the normal coordinate system (with respect to the connection (6.5)). We employ normal coordinates for both variables y, z in (5.16). With this adjustment, the integral looks like

$$f \star g \sim \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(v',v'')} L(v', v'') f(\exp_x(v')) g(\exp_x(v'')) \frac{\mathrm{d}v'\,\mathrm{d}v''}{(\pi\hbar)^{2n}}. \tag{6.6}$$

Here, the integration space, $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, is just $T_x(T^*\mathcal{M}) \times T_x(T^*\mathcal{M})$; $v'v''$ are normal coordinates and the phase function $S = \int_\Sigma \omega$, cf (5.14). The amplitude function L is given by (5.10) and (5.13): namely, $L = (\varphi/\mathcal{J})l \otimes l$, where l is the density of the Liouville measure on $T^*\mathcal{M}$ expressed in normal coordinates.

Lemma 10. *The following Taylor expansions hold*

$$S = S_2 + S_4 + O^6, \qquad L = 1 + L_2 + O^4, \qquad l = 1 + l_2 + O^4, \tag{6.7}$$

where S_2, L_2, l_2 are homogeneous polynomials of degree 2 and S_4 is of degree 4; the remainders O^4 and O^6 are of degrees 4 and 6 in normal coordinates near zero in $\mathbb{R}^{2n} \otimes \mathbb{R}^{2n}$. Formulae for the quadratic forms S_2 and L_2 are

$$S_2(v', v'') = 2\langle\Omega v'', v'\rangle, \qquad L_2(v', v'') = -\frac{1}{2}\langle\underline{\mathcal{R}}(\underline{v}' - \underline{v}''), \underline{v}' - \underline{v}''\rangle + l_2(v') + l_2(v''). \tag{6.8}$$

Here, Ω is the $2n \times 2n$ matrix of the symplectic form,

$$\Omega = \begin{bmatrix} F & I \\ -I & 0 \end{bmatrix},$$

$\underline{v} \in \mathbb{R}^n$ denotes the q-components of the vector $v \in \mathbb{R}^{2n} = \mathbb{R}^n_q \times \mathbb{R}^n_p$ and $\underline{\mathcal{R}}$ is the symmetric part of the $n \times n$ Ricci tensor on $\mathcal{M}$

$$\underline{\mathcal{R}}_{ls} \stackrel{\text{def}}{=} \tfrac{1}{2}\left(\underline{R}^k_{lks} + \underline{R}^k_{skl}\right), \tag{6.9}$$

where $\underline{R} = [\underline{\nabla}, \underline{\nabla}]$ is the curvature tensor (skew symmetric in the last pair of indices).

Proof. The first formula in (6.8) is obvious. Indeed, from (5.14), we claim that

$$S = \tfrac{1}{2}\Omega V''V' + O^4,$$

where V', V'' are the sides of the linearization of the triangle $\Sigma(x, y, z)$ at the point $x = (a, \xi)$. The sides V', V'' are twice as long as the normal coordinates v', v'' of the midpoints $y = (b, \eta), z = (c, \zeta)$ of this triangle. This is the reason for the factor of 2 in the first formula of (6.8).

To prove the second formula in (6.8), we first deduce from (5.10) and (5.13) that

$$L_2(v', v'') = l_2(v') + l_2(v'') + \alpha_2(\underline{v}', \underline{v}'') + \beta_2(\underline{v}', \underline{v}''). \tag{6.10}$$

Here, α_2 and β_2 are the second-order Taylor terms about the point a of the functions

$$\begin{aligned} \alpha &\stackrel{\text{def}}{=} 2^n |\det d(s_c s_b s_a)(Q)|^{1/2} \mathcal{J}^{-1}(Q), \\ \beta &\stackrel{\text{def}}{=} [j_a(Q) j_b(s_a(Q)) j_c(s_b s_a(Q))]^{1/2}, \end{aligned} \tag{6.11}$$

where Q is the solution of problem (5.11) in the neighbourhood of the origin point a.

The Taylor expansion of Q, in normal coordinates, results from

$$Q = \underline{\exp}_a(\underline{v}'' - \underline{v}' + O^2). \tag{6.12}$$

The Jacobi matrix of the mapping $s_c s_b s_a$ reads

$$d(s_c s_b s_a)(\underline{\exp}_a(\underline{v})) = -I + 2(\underline{v}'' - \underline{v}' - \underline{v})\underline{\Gamma} + O^2.$$

From here and (6.12), it follows that

$$d(s_c s_b s_a) = -I + O^2,$$

and so $\alpha = 1 + O^4$. This means that the second-order Taylor term of the function α at the point a is just zero: $\alpha_2 = 0$.

The Jacobians j_a, j_b, j_c which compose the function β in (6.11) are found in the beginning of section 3. Here, one has

$$j_q(\underline{\exp}_q(\underline{v})) = 1 - \tfrac{1}{3}\langle \underline{\mathcal{R}}\,\underline{v}, \underline{v}\rangle + O^4,$$

where $\underline{\mathcal{R}}$ is defined by (6.9). Therefore,

$$\beta = 1 - \tfrac{1}{2}\langle \underline{\mathcal{R}}(\underline{v}'' - \underline{v}'), \underline{v}'' - \underline{v}'\rangle + O^4.$$

Combining this with (6.10), we conclude that the second formula in (6.8) holds. □

Lemma 11. *The fourth-degree contribution S_4 in expansion (6.7) satisfies the following estimate:*

$$\langle \partial', \Psi\partial''\rangle S_4(v', v'') = O^2(\underline{v}') + O^2(\underline{v}''). \tag{6.13}$$

Proof. The exponential map corresponding to Christoffel symbols (6.5*a*) has the Taylor expansion

$$\exp_{a,\xi}(v) = \begin{pmatrix} a + \underline{v} - \tfrac{1}{2}\underline{\Gamma}(a)\underline{v}\,\underline{v} + \tfrac{1}{6}B(a)\underline{v}\,\underline{v}\,\underline{v} + O(\underline{v}^4) \\ \xi + \underline{\underline{v}} + \underline{\Gamma}(a)\underline{v}\,\underline{\underline{v}} - \tfrac{1}{2}(\xi B(a) + C(a))\underline{v}\,\underline{v} + O(\underline{v}^3) + O(\underline{v}^2\underline{\underline{v}}) \end{pmatrix},$$

where the matrices B, C are defined in (6.5b), and by $\underline{v}$ we indicate the p-component of the vector v.

From this formula and (5.4), we derive the following expression for the fourth-degree component S_4 of the phase function:

$$S_4(v', v'') = 2\langle \underline{\Gamma}\, \underline{v}'\underline{v}'', \underline{\Gamma}\, \underline{v}'\underline{v}''\rangle - 2\langle \underline{\Gamma}\, \underline{v}''\underline{v}'', \underline{\Gamma}\, \underline{v}'\underline{v}''\rangle + O(\underline{v}'\underline{v}''^3) \\ + O(\underline{v}''\underline{v}''^3) + O(\underline{v}'\underline{v}'^2\underline{v}'') + O(\underline{v}''\underline{v}''^2\underline{v}') + O^3(\underline{v}', \underline{v}''). \tag{6.14}$$

Here, the remainders do not contribute to the second-order derivatives in $\langle \partial', \Psi\partial''\rangle S_4$ and so we do not need to know their explicit expression. The first two terms in (6.14) provide formula (6.13). $\square$

Now we make the rescaling $v = \sqrt{\hbar}u$ in the integrand of (6.6) and use lemma 10 to get

$$f \star g \sim \iint \mathrm{e}^{\mathrm{i}S_2(u',u'')}[1 + \hbar(\mathrm{i}S_4 + L_2) + O(\hbar^2)](u', u'') \\ \times f(\exp_x(\sqrt{\hbar}u'))g(\exp_x(\sqrt{\hbar}u''))\frac{\mathrm{d}u'\,\mathrm{d}u''}{\pi^{2n}}. \tag{6.15}$$

By expanding the exponential mappings in (6.15), we reduce the calculation of the asymptotics of $f \star g$ to the evaluation of simple integrals like

$$\frac{1}{\pi^{2n}}\iint \mathcal{P}_{2m}(u', u'')\exp\{2\mathrm{i}\langle \Omega u', u''\rangle\}\,\mathrm{d}u'\,\mathrm{d}u'' \\ = \sum_{r=0}^{m}\frac{(-i)^r}{2^r r!}\sum_{1\leqslant l_j,s_j\leqslant 2n}\Psi^{l_1s_1}\cdots\Psi^{l_rs_r}\big(\partial'_{l_1}\cdots\partial'_{l_r}\partial''_{s_1}\cdots\partial''_{s_r}\mathcal{P}_{2m}\big)(0,0), \tag{6.16}$$

with some polynomials, $\mathcal{P}_{2m}$, of degree $2m$, where $m = 0, 1, \ldots$.

In order to know the kth term in expansion (6.1), one must take into account contributions of integrals like (6.16) for degrees $2m \leqslant 2k$. In this way, from (6.15), we obtain the expansion (6.3) and formula (6.4) for the operator G_2, where the tensor $\mathcal{B}$ is given by

$$\mathcal{B}^{sl} = 2\Psi^{sm}\Psi^{rl}\partial'_m\partial''_r L_2 + \Psi^{sm}\Psi^{rl}\Psi^{jk}\partial'_m\partial'_j\partial''_r\partial''_k S_4. \tag{6.17}$$

The second formula in (6.8) implies that

$$\partial'\partial'' L_2 = \begin{pmatrix}\underline{\mathcal{R}} & 0\\ 0 & 0\end{pmatrix},$$

where $\underline{\mathcal{R}}$ is given by (6.9). From (6.13), one concludes that the second term in (6.17) vanishes. Therefore,

$$\mathcal{B} = 2\Psi\cdot\begin{pmatrix}\underline{\mathcal{R}} & 0\\ 0 & 0\end{pmatrix}\cdot\Psi. \tag{6.18}$$

Proposition 5. *The curvature tensor R of the magneto-geodesic connection ∇ on $T^*\mathcal{M}$ is given by*

$$R^{q^s}_{q^iq^jq^k} = \underline{R}^s_{ijk},$$

$$R^{p_s}_{p_jq^iq^k} = \underline{R}^j_{ski}, \qquad R^{p_s}_{q^ip_jq^k} = -R^{p_s}_{q^iq^kp_j} = \tfrac{1}{3}\big(\underline{R}^j_{ski} + \underline{R}^j_{iks}\big),$$

$$R^{p_s}_{q^iq^jq^k} = \tfrac{1}{3}\mathop{\mathfrak{S}}_{is} p_m\big(\underline{\nabla}_s\underline{R}^m_{ikj} + 3\underline{\Gamma}^m_{sl}\underline{R}^l_{ijk} - \underline{\Gamma}^m_{jl}\underline{R}^l_{iks} + \underline{\Gamma}^m_{kl}\underline{R}^l_{ijs}\big) + M_{sijk},$$

where

$$M_{sijk} \overset{\text{def}}{=} \tfrac{1}{3}\big(\underline{\nabla}_i \underline{\nabla}_s F_{jk} + 2\underline{R}^l_{ijk} F_{ls} + \underset{ijk}{\mathfrak{S}}\, \underline{R}^l_{sij} F_{lk}\big) \qquad (\textit{magnetic curvature}).$$

Here, $\underline{R} = \underline{R}(q)$ *and* $F = F(q)$ *are the curvature of the connection* $\underline{\nabla}$ *and the magnetic tensor on* $\mathcal{M}$*. All other components of the curvature* R *vanish.*

We first remark that the block R^p_{qqq} of the curvature tensor R is not itself a tensor. But the part M, which we call the *magnetic curvature*, is a tensor on $\mathcal{M}$. This part of the total curvature entangles the magnetic field F with the curvature tensor $\underline{R}$ on $\mathcal{M}$. Also note that the only p-dependent part of the curvature is the R^p_{qqq} block (where p enters linearly); all other parts are strictly q dependent.

For any symplectic connection, the Ricci tensor

$$\mathcal{R}_{ij} \overset{\text{def}}{=} R^k_{ikj} \tag{6.19}$$

is symmetric [78]. Thus, for the magneto-geodesic connection on $T^*\mathcal{M}$, we have $\mathcal{R}_{ij} = \mathcal{R}_{ji}$.

Corollary 5. *The Ricci tensor for the magneto-geodesic connection on* $T^*\mathcal{M}$ *is given by*

$$\mathcal{R} = \frac{2}{3}\begin{pmatrix} \underline{\mathcal{R}} & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.20}$$

where $\underline{\mathcal{R}}$ *is the symmetric part of the Ricci tensor for the affine connection on* $\mathcal{M}$*.*

Now one just has to compare this formula for the Ricci tensor with formula (6.18) for the tensor $\mathcal{B}$ in the $\hbar$-expansion of the $\star$-products (6.3) and (6.4).

Altogether we have proved the following statement.

Theorem 2. *The magneto-geodesic product (5.16) over* $T^*\mathcal{M}$ *has the asymptotic expansion (6.3) with the second-order term* G_2 *given by formula (6.4). In this formula, the connection* ∇ *coincides with the magneto-geodesic connections (6.5), (6.5a), and the tensor* $\mathcal{B}$ *is*

$$\mathcal{B} = 3\Psi\mathcal{R}\Psi, \tag{6.21}$$

where $\mathcal{R}$ *is the Ricci tensor of this connection.*

The magneto-geodesic connection ∇ on $T^*\mathcal{M}$ entangles, on the phase space level, the original affine connection $\underline{\nabla}$ and the magnetic field (Faraday tensor) F on $\mathcal{M}$. In the case of zero magnetic field $F = 0$, this connection was obtained in [28] via a version of the deformation quantization approach by using a different star product which does not obey axioms (2.6).

The part of the Christoffel symbol in (6.5*a*) which describes the 'interaction' between the affine and magnetic structures on $\mathcal{M}$ is given by the tensor C. The tensor C contains the symmetrized covariant derivative of the magnetic tensor F. We label this part of the connection as the *magneto-geodesic coupling* tensor. The usual covariant derivative tensor $\underline{\nabla}F$ we call the *magnetic inhomogeneity*.

Let us examine the symmetric and skew-symmetric properties of these two tensors. We represent this pair of tensors with the notation

$$F_{[jk]l} \overset{\text{def}}{=} \underline{\nabla}_l F_{jk}, \qquad F_{j\{kl\}} \overset{\text{def}}{=} C_{jkl}.$$

Proposition 6. *(a) The magnetic inhomogeneity tensor $F_{[jk]l}$ is skew symmetric in the first pair of indices; the magneto-geodesic coupling tensor $F_{j\{kl\}}$ is symmetric in the last pair of indices. They are related to each other by the following duality formulae:*

$$F_{j\{kl\}} = \tfrac{1}{3}(F_{[jk]l} + F_{[jl]k}), \qquad F_{[jk]l} = F_{j\{kl\}} - F_{k\{jl\}}.$$

(b) Both of these tensors obey the cyclic property

$$\underset{jkl}{\mathfrak{S}} F_{j\{kl\}} = 0, \qquad \underset{jkl}{\mathfrak{S}} F_{[jk]l} = 0. \tag{6.22}$$

(c) The magneto-geodesic coupling tensor is related to the magnetic curvature tensor M by

$$\underline{\nabla}_j F_{s\{ik\}} - \underline{\nabla}_k F_{s\{ij\}} = M_{isjk}.$$

(d) On Riemannian manifolds $\mathcal{M}$ (with the Levi–Civita connection $\underline{\nabla}$ defined by metric g), the current covector

$$j_s = \tfrac{2}{3} F_{s\{ik\}} g^{ik}$$

satisfies the continuity equation $g^{ls}\underline{\nabla}_l j_s = 0$ and generates the exact 2-form $\varkappa \equiv dj$ with coefficients

$$\varkappa_{sm} = g^{ik}\big(3M_{iksm} + 2F_{il}\underline{R}^{l}_{ksm}\big).$$

The identity $\mathrm{d}\varkappa = 0$, or $\mathfrak{S}\nabla_r \varkappa_{sm} = 0$, ties together the magnetic curvature M and the Riemannian curvature $\underline{R}$ on $\mathcal{M}$.

The last statement in (6.22) is just the homogeneous Maxwell equation for the magnetic field (that ensures that there are no Dirac monopoles) and the first identity in (6.22) is the 'dual' to the Maxwell equation and incorporates the magneto-geodesic coupling tensor.

Note that within the class of tensors obeying the condition $\mathfrak{S}C_{jkl} = 0$, formulae (6.5*a*) and (6.5*b*) determine a unique connection on $T^*\mathcal{M}$ having the property that the symplectic form is covariantly constant, $\nabla\omega = 0$.

In the Euclidean case $\mathcal{M} = \mathbb{R}^n$, the magneto-geodesic connection ∇ coincides with the magnetic connection found in [5]. In this case, the curvature tensor R of ∇ (see proposition 5) is reduced to the magnetic curvature which is just the second derivative matrix $M_{sijk} = \frac{1}{3}D^2_{si}F_{jk}$. Thus, the curvature R is constant iff the field F is quadratic in Euclidean coordinates. In this way, we obtained in [5] an example of a symmetric symplectic space which can be explicitly and exactly quantized.

It should be interesting to study the equation

$$\nabla R = 0 \tag{6.23}$$

for the general magneto-geodesic connection. Under condition (6.23), one again has an example of a symmetric symplectic structure, now on $T^*\mathcal{M}$, which is explicitly and exactly (not formally and not asymptotically) quantized. It would be interesting to compare this construction of quantized symmetric spaces with [79, 80].

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